PROPAGATION OF HARMONIC WAVES IN AN ELASTIC ROD OF ELLIPTICAL CROSS-SECTION

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ABSTRACT

Using the potential equations of motion of linear elasticity, the propagation of harmonic waves in an infinite rod of elliptical cross-section is investigated. The frequency equations for the propagation of flexural waves in rods with (i) zero surface displacements, and (ii) zero surface stresses are obtained in the form of infinite determinants, the elements of which involve Mathieu functions and their derivatives. It is shown that these determinants can be written in diagonal form when the eccentricity goes to zero and in the light of this possible numerical procedures are discussed for small values of the eccentricity.

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1.1 INTRODUCTION

The equations of motion of a linear elastic solid can be expressed in terms of potentials, which satisfy scalar wave equations. In the case of axial symmetry two independent potentials are involved, whereas in the event of non-axially symmetric motions, three independent potentials arise. The former case is well known and has been exploited in many areas, but the latter case seems to have received very little attention in the elastodynamics literature (however the analogous case in electrodynamics has been thoroughly discussed). One of the purposes of this thesis is to illustrate how the use of the potential equations of motion leads to more systematic solution procedures to non-axially symmetric problems, and this is shown in detail in connection with the derivation of the frequency equation for the propagation of harmonic flexural waves in an infinite, circular, elastic rod. The main portion of the thesis is concerned with the propagation of harmonic waves in rods of elliptical cross-section, using the linear equations of elasticity in potential form. The separation solutions of these equations, for the geometry at hand, are in the form of products of Mathieu functions and modified Mathieu functions and these solutions are applied to two physical problems, namely, the infinite elliptical rod with zero surface displacements, and the infinite elliptical rod with zero surface stresses. In each of these cases it is found that an infinite superposition of the separation solutions is necessary. It is further found that in both cases three basic modes of motion exist, corresponding to compressional, flexural, and torsional waves in a circular rod. The frequency equations in all of the above cases are in the form of infinite determinants and those corresponding to a "flexural" mode of vibration have been examined in some detail. It is found that when the elliptical rod becomes circular, the infinite determinants involved can be written in a diagonal form, the elements of which are the frequency equations for the propagation of flexural and higher circumferential modes in a circular rod. Finally, some discussion as to possible numerical procedures is given.

1.2 DEVELOPMENT OF POTENTIAL EQUATIONS OF MOTION

The equations of motion of a homogeneous elastic body are given by (Love [1])*

$$(\lambda + 2\mu)\nabla(\nabla \cdot \vec{u}) - \mu\nabla \times \nabla \times \vec{u} - \rho \frac{\partial^2 \vec{u}}{\partial t^2} + \vec{F} - \frac{\alpha E}{1-2\nu}\nabla T = 0 \quad (1)$$

when $\vec{u}$ is the displacement vector, $\vec{F}$ is the body force per unit volume, T is the temperature, λ and μ are Lamē's constants, E is Young's modulus, ν is Poisson's ratio, ρ is the density, and α is the thermal coefficient of linear expansion. The dilatation Δ , and rotation $\vec{\Omega}$, are given by

$$\Delta = \nabla \cdot \vec{u} \quad (2)$$

$$2\,\vec{\Omega} = \nabla \times \vec{u} \quad (3)$$

and in terms of these quantities (1) may be written

$$(\lambda + 2\mu)\nabla\Delta - \mu\nabla \times 2\vec{\Omega} - \rho \frac{\partial^2 \vec{u}}{\partial t^2} + \vec{F} - \frac{\alpha E}{1-2\nu}\nabla T = 0 \quad (4)$$

Taking the divergence and curl of (4) gives, on noting from (3) that div $\vec{\Omega} = 0$,

$$(\lambda + 2\mu)\nabla^2\Delta - \rho \frac{\partial^2 \Delta}{\partial t^2} + \nabla \cdot \vec{F} - \frac{\alpha E}{1-2\nu}\nabla^2 T = 0 \quad (5)$$

$$\mu\nabla^2 2\vec{\Omega} - \rho \frac{\partial^2 2\vec{\Omega}}{\partial t^2} + \nabla \times \vec{F} = 0 \quad (6)$$

*Numbers in brackets designate References at end of text.

The displacement vector may be written in terms of a scalar potential φ and a vector potential $\vec{A}$ (Lame's potentials) by means of (see Sternberg [2] for a detailed analysis).

$$\vec{u} = \nabla \varphi + \nabla \times \vec{A} \tag{7}$$

The left hand side of this expression involves three unknowns, whereas the right hand side contains four, namely, φ and the three components of $\vec{A}$. Thus another condition must be imposed on $\vec{A}$, but this condition must be such that the field quantity $\vec{u}$ remains the same, i.e., the field quantities must be gauge invariant. The forms of this condition will be discussed later.

Substituting (7) into (2) and (3) gives

$$\Delta = \nabla^2 \varphi \tag{8}$$

$$2\vec{\Omega} = \nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{9}$$

Assuming that the body force $\vec{F}$ is given by

$$\vec{F} = \nabla \Theta + \nabla \times \vec{B} , \tag{10}$$

substitution of (8) and (10) into (5) gives, as the scalar potential equation of motion

$$\nabla^2 [(\lambda + 2\mu)\nabla^2\varphi - \rho \frac{\partial^2\varphi}{\partial t^2} + \Theta - \frac{\alpha E}{1-2\nu} T] = 0$$

A sufficient condition for the satisfaction of this equation is

$$\nabla^2 \varphi = \frac{1}{c_d^2} \frac{\partial^2\varphi}{\partial t^2} - \frac{\Theta}{\lambda+2\mu} + \frac{\alpha E}{(\lambda+2\mu)(1-2\nu)} T \tag{11}$$

where $c_d^2 = (\lambda + 2\mu)/\rho$ is the dilatational wave speed squared.

Substituting (9) and (10) into (6), gives, as the vector potential equation of motion

$$\mu\nabla^2(\nabla\times\nabla\times\vec{A}) - \rho\frac{\partial^2}{\partial t^2}(\nabla\times\nabla\times\vec{A}) + \nabla\times\nabla\times\vec{B} = 0$$

which may be written

$$\nabla\times\nabla\times[\mu\nabla^2\vec{A} - \rho\frac{\partial^2\vec{A}}{\partial t^2} + \vec{B}] = 0$$

A sufficient condition for the satisfaction of this is that

$$\nabla^2\vec{A} = \frac{1}{c_s^2}\frac{\partial^2\vec{A}}{\partial t^2} - \frac{1}{\mu}\vec{B} \tag{12}$$

where $c_s^2 = \mu/\rho$ is the shear wave speed squared.

From this it appears that the divergence of $\vec{A}$ is arbitrary. Consider however the case of $\vec{B} = 0$. Taking the divergence of (9) gives

$$\text{div } 2\vec{\Omega} = 0 = \nabla^2(\nabla\cdot\vec{A}) - \nabla\cdot(\nabla^2\vec{A})$$

showing that in this case the operators ∇^2 and $\nabla\cdot$ are commutative. Hence taking the divergence of (12) leads to

$$(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2})(\nabla\cdot\vec{A}) = 0 \tag{13}$$

Thus it is seen that the divergence of $\vec{A}$ is not arbitrary but must satisfy one of the conditions:

(i) $$\nabla\cdot\vec{A} = 0 \tag{14}$$

(ii) $$\nabla^2(\nabla\cdot\vec{A}) - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}(\nabla\cdot\vec{A}) = 0 \tag{15}$$

Since $\vec{A}$ must satisfy either (14) or (15), in effect there are only two independent components of the vector potential. In general cylindrical and spherical coordinates it is possible to choose these components such that they satisfy scalar wave equations. This fact is well known in electromagnetic theory (see, for instance, Morse and Feshback [3], Chapter 13), but does not appear to have been systematically exploited in elastodynamics. As mentioned in the introduction, one of the purposes of this section is to illustrate how the present approach leads to more systematic solution procedures.

It is shown in Appendix A that in general cylindrical coordinates* the vectors

$$\vec{A} = \psi \vec{e_z} + \nabla \times (\chi \vec{e_z}) \tag{16}$$

$$\vec{A} = \nabla \times (\psi \vec{e_z}) + \nabla \times \nabla \times (\chi \vec{e_z}) \tag{17}$$

where ψ and χ are scalar functions, and $\vec{e_z}$ is a unit vector in the z-direction, satisfy the vector wave equation (12), with $\vec{B} = 0$, provided ψ and χ satisfy the scalar wave equations

$$\nabla^2 \psi = \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} \tag{18}$$

$$\nabla^2 \chi = \frac{1}{c_s^2} \frac{\partial^2 \chi}{\partial t^2} \tag{19}$$

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*By general cylindrical coordinates, is meant the coordinates x_1, x_2, x_3 where $z = x_3$ $x + i y = f(x_1 + ix_2)$, where f is an analytic function of $(x_1 + ix_2)$.

The nature of ψ and χ can be seen in the case of axially-symmetric motions in circular cylindrical coordinates. In that case (16) becomes

$$\vec{A} = \psi \,\vec{e_z} - \frac{\partial\chi}{\partial r}\, \vec{e_\theta}$$

where $\vec{e_\theta}$ is a unit vector in the θ-direction.

This should be compared with the more usual approach to axially-symmetric problems, namely, (i) for the torsional motion, where $\vec{A} = \psi\, \vec{e_z}$ (ii) for non-torsional motion (for example, compressional wave motion in a circular rod) where

$$\vec{A} = -\frac{\partial\chi}{\partial r}\,\vec{e_\theta} \, .$$

Hence, it is seen that ψ represents torsional type motion and χ represents non-torsional type motion.

It can be shown, with use of (19), that in the case of axial symmetry (17) becomes,

$$\vec{A} = \frac{\partial^2\chi}{\partial r\partial z}\,\vec{e_r} - \frac{\partial\psi}{\partial r}\,\vec{e_\theta} + \left(\frac{\partial^2\chi}{\partial z^2} - \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}\right)\vec{e_z}$$

In this case, ψ represents the non-torsional motion and χ represents the torsional motion.

Note that for the choice of $\vec{A}$ given by (17), $\nabla \cdot \vec{A} = 0$, so that (14) is satisfied, whereas the choice given by (16) is such that $\nabla \cdot \vec{A} = \frac{\partial\psi}{\partial z}$ and (15) is satisfied, provided (18) and (19) hold.

In spherical coordinates the vector

$$\vec{A} = \nabla \times (r\,\psi\,\vec{e_r}) + \nabla \times \nabla \times (r\,\chi\,\vec{e_r}) \tag{20}$$

where ψ and χ are scalar functions, and $\vec{e_r}$ is a unit vector in the r-direction, satisfy the vector wave equation, provided

$$\nabla^2 \psi = \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2}$$

$$\nabla^2 \chi = \frac{1}{c_s^2} \frac{\partial^2 \chi}{\partial t^2}$$

It is interesting to note that in the case of spherical coordinates only the choice $\nabla \cdot \vec{A} = 0$ is available, since no analogue of (16) appears to exist.

Some applications involving cylindrical coordinate systems of these equations will now be discussed.

1.3 CIRCULAR CYLINDRICAL COORDINATES

Here some problems involving an infinite, circular cylindrical rod will be discussed. The z-axis of the circular cylindrical coordinate system is taken to be along the rod axis. Taking $\vec{A}$ as given by (16), the displacement components in terms of the potentials are, from (7),

$$u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \chi}{\partial z \partial r} \tag{21}$$

$$u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial^2 \chi}{\partial z \partial \theta} \tag{22}$$

$$u_z = \frac{\partial \varphi}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \chi}{\partial r}\right) - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \tag{23}$$

If $\vec{A}$ as given by (17) is taken instead of (16), it can be shown that the effect in (21), (22), and (23), is to replace ψ by $-\frac{1}{c_d^2}\frac{\partial^2\chi}{\partial t^2}$ and χ by ψ. Thus it is seen that there is no appreciable algebraic advantage in choosing one form of the vector potential over the other. [This is illustrated by taking $\vec{A}$ as given by (17) in the elliptical rod case.]

Relations of the form of (21), (22) and (23) were given by Harkrider [4] in connection with problems involving non-axially symmetric sources in a layered medium, and by Baltrukonis, Gottenberg, and Shreiner, [5] in connection with a problem involving a cylindrical shell, but these authors offer no explanation as to their origin. The pertinent stress-strain relations for the problems at hand are

$$\sigma_{rr} = \lambda \nabla + 2\mu \frac{\partial u_r}{\partial r} \tag{24}$$

$$\sigma_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{1}{r}u_\theta\right)\right] \tag{25}$$

$$\sigma_{rz} = \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] \tag{26}$$

where

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{27}$$

Assuming harmonic wave trains propagating in the z-direction, the separation solutions of (11), (18), and (19), with $\Theta = T = \vec{B} = 0$, can readily be shown to be

$$\varphi(r,\theta,z,t) = [B_1 J_n(p_1 r) + B_2 Y_n(p_1 r)] \cos n\,(\theta-\theta_1)e^{i(kz-wt)} \tag{28}$$

$$\chi(r,\theta,z,t) = [D_1 J_m(p_2 r) + D_2 Y_m(p_2 r)] \cos m\,(\theta-\theta_2)e^{i(kz-wt)} \tag{29}$$

$$\psi(r,\theta,z,t) = [H_1 J_\ell(p_2 r) + H_2 Y_\ell(p_2 r)] \cos \ell\,(\theta-\theta_3)e^{i(kz-wt)} \tag{30}$$

where
$$p_1^2 = \frac{\omega^2}{c_d^2} - k^2 \tag{31}$$

$$p_2^2 = \frac{\omega^2}{c_s^2} - k^2 \tag{32}$$

k is the wavenumber, ω the frequency, B_1, B_2, D_1, D_2, H_1, H_2, θ_1, θ_2, and θ_3 are arbitrary constants, n, m, and ℓ, are integers (for single-valued solutions), and J and Y denote Bessel functions of the first and second kind, respectively.

A specific problem will now be considered, namely, the determination of the frequency equation (i.e., the ω-k equation) describing the propagation of harmonic flexural waves in an infinite, solid, circular rod, with stress-free boundaries. In this case the following values of some of the arbitrary constants are required: (i) $n = m = \ell = 1$ (so that each of the displacements has one nodal plane intersecting the rod axis —— a characteristic feature of flexure), (ii) $\theta_1 = \theta_2 = 0 \quad \theta_3 = \frac{\pi}{2}$, (iii) $B_2 = D_2 = H_2 = 0$, for finiteness at $r = 0$. With these choices of constants, substitution of (28), (29), and (30), into (21), (22), and (23), gives

$$u_r = \left\{ \begin{aligned} & B_1[p_1 J_o(p_1 r) - \frac{1}{r} J_1 (p_1 r)] \\ & + H_1 \frac{1}{r} J_1 (p_2 r) \\ & + ikD_1[p_2 J_o(p_2 r) - \frac{1}{r} J_1 (p_2 r)] \end{aligned} \right\} \cos \theta \, e^{i(kz-wt)} \tag{33}$$

$$u_\theta = -\left\{ B_1 \frac{1}{r} J_1 \ (p_1 r) \right.$$

$$+ H_1 \ [p_2 J_o (p_2 r) - \frac{1}{r} J_1 \ (p_2 r)] \qquad (34)$$

$$\left. + ik \ D_1 \frac{1}{r} J_1 \ (p_2 r) \right\} \sin\theta \ e^{i(kz-wt)}$$

$$u_z = [ik \ B_1 \ J_1 \ (p_1 r) + p_2^2 \ D_1 \ J_1 \ (p_1 r)] \cos\theta \ e^{i(kz-wt)} \qquad (35)$$

Substituting (33), (34), and (35), into (24), (25), (26), and (27), gives

$$\sigma_{rr} = \left\langle B_1 \left\{ [2\mu(\frac{2}{r^2} - p_1^2) - \lambda \frac{\omega^2}{c_d^2}] \ J_1(p_1 r) - \frac{2\mu p_1}{r} J_o \ (p_1 r) \right\} \right.$$

$$+ 2\mu \ H_1 \ [\frac{p_2}{r} J_o \ (p_2 r) - \frac{2}{r^2} J_1 \ (p_2 r)] \qquad (36)$$

$$\left. + 2\mu ik \ D_1 \ [(\frac{2}{r^2} - p_2^2) \ J_1 \ (p_2 r) - \frac{p_2}{r} J_o \ (p_2 r)] \right\rangle \cos\theta \ e^{i(kz-wt)}$$

$$\sigma_{r\theta} = \mu \left\{ 2B_1 \ [\frac{2}{r^2} J_1 \ (p_1 r) - \frac{p_1}{r} J_o \ (p_1 r)] \right.$$

$$+ H_1 [(p_2^2 - \frac{4}{r^2}) \ J_1 \ (p_2 r) + \frac{2p_2}{r} J_o \ (p_2 r)] \qquad (37)$$

$$\left. + 2ik \ D_1 [\frac{2}{r^2} J_1 \ (p_2 r) - \frac{p_2}{r} J_o \ (p_2 r)] \right\} \sin\theta \ e^{i(kz-wt)}$$

$$\sigma_{rz} = \mu \left\{ 2ik \ B_1 \ [p_1 \ J_o \ (p_1 r) - \frac{1}{r} J_1 \ (p_1 r)] \right.$$

$$+ ik \ H_1 \frac{1}{r} J_1 \ (p_2 r) \qquad (38)$$

$$\left. + \left(\frac{\omega^2}{c_s^2} - 2 \ k^2\right) D_1 \ [p_2 \ J_o \ (p_2 r) - \frac{1}{r} J_1 \ (p_2 r)] \right\} \cos\theta \ e^{i(kz-wt)}$$

The boundary conditions for the present problem are

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0, \ r = a$$

where a is the rod radius. Applying these to (36), (37), and (38) gives three homogeneous algebraic equations in the three unknowns B_1, H_1, and D_1. For these equations to have a solution, the determinant of the coefficients must equal zero, which yields the frequency equation. Going through this procedure one obtains, after some algebra, the frequency equation (see, for instance, Bancroft [6]).

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = 0 \tag{39}$$

where

$$\alpha_{11} = \left[2\mu\left(\frac{2}{a^2} - p_1^2\right) - \lambda \frac{\omega^2}{c_d^2}\right] J_1\ (p_1 a) - \frac{2\mu p_1}{a} J_o\ (p_1 a)$$

$$\alpha_{12} = 2\mu \left[\frac{p_2}{a} J_o\ (p_2 a) - \frac{2}{a^2} J_1\ (p_2 a)\right]$$

$$\alpha_{13} = 2\mu i k \left[\left(\frac{2}{a^2} - p_2^2\right) J_1\ (p_2 a) - \frac{p_2}{a} J_o\ (p_2 a)\right]$$

$$\alpha_{21} = 2\left[\frac{2}{a^2} J_1\ (p_1 a) - \frac{p_1}{a} J_o\ (p_1 a)\right]$$

$$\alpha_{22} = \left[\left(p_2^2 - \frac{4}{a^2}\right) J_1\ (p_2 a) + \frac{2p_2}{a} J_o\ (p_2 a)\right]$$

$$\alpha_{23} = 2ik\left[\frac{2}{a^2} J_1\ (p_2 a) - \frac{p_2}{a} J_o\ (p_2 a)\right]$$

$$\alpha_{31} = 2ik\left[p_1\ J_o\ (p_1 a) - \frac{1}{a} J_1\ (p_1 a)\right]$$

$$\alpha_{32} = ik \frac{1}{a} J_1 \ (p_2 a)$$

$$\alpha_{33} = \left(\frac{\omega^2}{c_s^2} - 2k^2 \right) \left[p_2 \ J_o \ (p_2 a) - \frac{1}{a} J_1 \ (p_2 a) \right]$$

The advantage of the present method lies in the systematic derivation of the displacements, (33), (34) and (35). In other approaches to this problem, such as, for example, Love's presentation of Pochammer's work ([1], pp. 291-292), these equations are presented as solutions to the displacement equations of motion, but they were deduced in an indirect manner. The present approach can also be utilized to give a more systematic derivation of the frequency equations of other problems of this nature, such as the cylindrical shell problem, as treated by Gazis* [7]. In particular it could be applied with some advantage to the problem of non-axially symmetric waves in a layered rod, a problem which does not seem to have received any attention in the literature, though the symmetric layered rod problem has been treated by McNiven, Sackman and Shark [8].

1.4 NON-CIRCULAR CYLINDRICAL COORDINATES

For cylindrical boundaries other than circular, elastodynamic problems are considerably more complicated and only very restricted information is available. Fox and Mindlin [9] have given a set of discrete points of the frequency spectrum for an infinite rectangular rod

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*It should be noted that the comment of Gazis that the divergence of $\vec{A}$ can be arbitrarily specified appears to be in error, in the light of equations (14) and (15). It can be shown that div $\vec{A}$ in his work satisfies the scalar wave equation, and so falls under the category given by (15).

with stress-free boundaries, for special values of the ratio of width to depth. As pointed out by Mindlin [10], the frequency spectrum in this case cannot be expressed in terms of a finite number of known transcendental functions, because of the complexities arising from mode conversion at the two perpendicular boundaries. Mathematically these difficulties could be anticipated because of the presence of two characteristic lengths in the problem (the square rod is a degenerate case), and are not of course confined to the rectangular rod. They arise whenever one has to satisfy nonmixed conditions (i.e., all stresses specified) at two or more perpendicular boundaries. Other work on the rectangular rod has been done by Morse [11], who satisfies the condition on one boundary exactly, and on the other approximately, and compares the results with experiment. Jones and Ellis [12] used plane stress theory to discuss pulse propagation in wide rectangular bars, theoretically and experimentally. Some numerical results have been obtained by Kynch and Green [13], using a perturbation technique, treating the circular cylinder as the basis for their perturbations.

Interest in the present section is in problems involving elliptical boundaries and here the available literature is even more restricted. However, some exact theory work in this area has been given. The recent work of Rosenfeld and Miklowitz [14], using the linear equations of motion, could be used to study the propagation of low frequency, small wavenumber, waves in an elliptical rod. Banaugh and Goldsmith [15], using an integral equation technique and numerical evaluation, studied the scattering of plane waves by elliptical inclusions in an infinite medium, and gave numerical results for the case of a compressional wave incident on a rigid inclusion. Work on the acoustic problem has also received some attention. Barakat [16], using a separation of variables technique, treated the scattering of plane waves from an elliptical cylinder for both Dirichlet and Neumann boundary conditions. Levy [17] also studied the scattering of acoustic waves from an elliptic cylinder, using the Keller theory of diffraction as well as a separation of variables technique.

Some approximate work has also been contributed. Kynch and Green [13] treated the elliptical rod as one of the cases in their perturbation technique and Kynch [18] later used the Rayleigh-Ritz technique to obtain further numerical information (this paper also contains an account of other elementary approximate theory approaches). Very recently Callahan ([19] and [20]) has used the Mindlin equations of motion to study the flexural vibrations of elliptical plates and rings, subject to various types of boundary conditions. The frequency equations he obtained are in the form of infinite determinants, and no numerical work was attempted (for work on elliptical plates involving classical theory, see McLachlan [21], pp. 309-312).

Here the problem of harmonic wave propagation in an infinite rod of elliptical cross-section will be solved, using the exact theory equations of motion.

Elliptic coordinates x_1, x_2, x_3, are defined in relation to Cartesian coordinates x, y, z, by

$$\begin{aligned} x &= f \cosh x_1 \cos x_2 \\ y &= f \sinh x_1 \sin x_2 \\ z &= x_3 \end{aligned} \tag{40}$$

where f is a constant (half of the length of the focal line of the ellipse). To specify the boundary of the ellipse (the rod surface), x_1 is set equal to a constant and the geometrical characteristics of this ellipse are (Fig. I).

$$\left.\begin{aligned} a &= f \cosh x_1 \\ b &= f \sinh x_1 \\ e &= \frac{\sqrt{a^2 - b^2}}{a} = \frac{f}{a} = \frac{1}{\cosh x_1} \end{aligned}\right\} \tag{41}$$

where a, b, and e, are the semi-major axis, semi-minor axis, and eccentricity, respectively. In terms of these coordinates, the wave equation (11) (with $\Theta = T = 0$) is (Moon and Spencer [22])

$$\frac{1}{h^2}\left(\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2}\right) + \frac{\partial^2\varphi}{\partial x_3^2} = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2} \tag{42}$$

where the metric coefficient h is given by

$$h^2 = \frac{f^2}{2}(\cosh 2x_1 - \cos 2x_2) \tag{43}$$

In this case (17) is chosen as the vector potential and in terms of elliptic coordinates it is given by

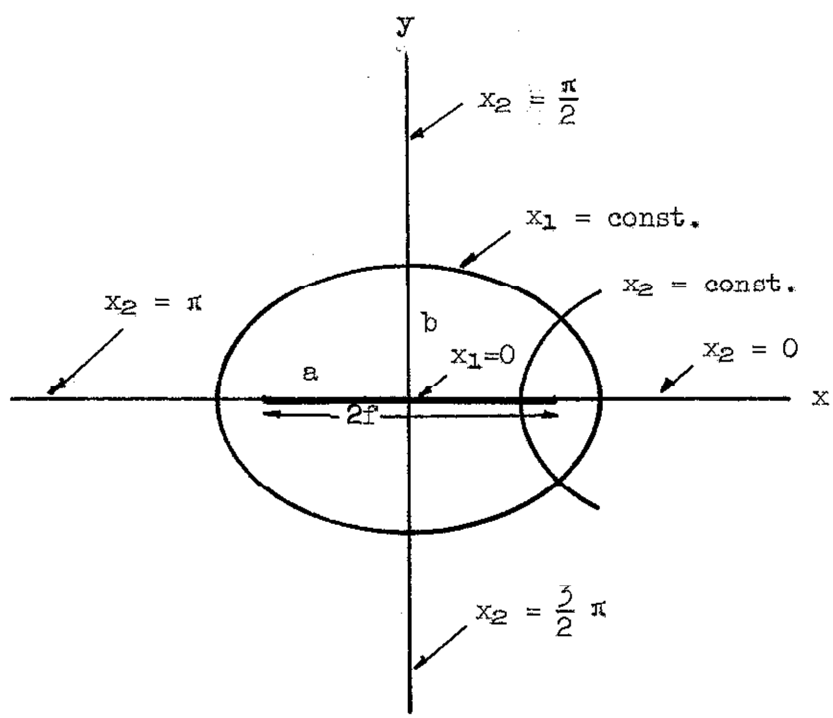

Fig. I. Geometry of Rod.

$$\vec{A} = \nabla \psi \times \overrightarrow{e_{x_3}} + \nabla\left(\frac{\partial X}{\partial x_3}\right) - \frac{1}{c_s^2}\frac{\partial^2 X}{\partial t^2}\overrightarrow{e_{x_3}} \tag{44}$$

where $$\nabla = \frac{\overrightarrow{e_{x_1}}}{h}\frac{\partial}{\partial x_1} + \frac{\overrightarrow{e_{x_2}}}{h}\frac{\partial}{\partial x_2} + \overrightarrow{e_{x_3}}\frac{\partial}{\partial x_3}$$

$\overline{e_{x_j}}$, $j = 1,2,3$, denotes a unit vector in the x_j-direction, and use has been made of the fact that X satisfies (19) in arriving at (44). Substituting (44) into (7), and using (18), gives for the displacements

$$u_1 = \frac{1}{h}\left(\frac{\partial\varphi}{\partial x_1} + \frac{\partial^2\psi}{\partial x_1 \partial x_3} - \frac{1}{c_s^2}\frac{\partial^3 X}{\partial t^2 \partial x_2}\right) \tag{45}$$

$$u_2 = \frac{1}{h}\left(\frac{\partial\varphi}{\partial x_2} + \frac{\partial^2\psi}{\partial x_2 \partial x_3} + \frac{1}{c_s^2}\frac{\partial^3 X}{\partial t^2 \partial x_1}\right) \tag{46}$$

$$u_3 = \frac{\partial\varphi}{\partial x_3} - \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x_3^2} \tag{47}$$

The pertinent stress-displacement relations are (see for instance, (48.7), (48.9), (48.10), and problem 1, p. 185, Sokolnikoff [23].)

$$\sigma_{11} = \lambda\,\Delta + 2\mu\left(\frac{1}{h}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h^2}\frac{\partial h}{\partial x_2}\right) \tag{48}$$

$$\sigma_{12} = \mu\left[\frac{\partial}{\partial x_1}\left(\frac{1}{h}u_2\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{h}u_1\right)\right] \tag{49}$$

$$\sigma_{13} = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{1}{h}\frac{\partial u_3}{\partial x_1}\right) \tag{50}$$

where $$\Delta = \nabla \cdot \vec{u} = \nabla^2\varphi \tag{51}$$

Substituting (45), (46) and (47) into (48), (49) and (50), gives, on using (51), and after some rearranging,

$$\sigma_{11} = \lambda \nabla^2 \varphi + 2\mu \left[\frac{1}{h} \frac{\partial}{\partial x_1} \left(\frac{1}{h} \frac{\partial \varphi}{\partial x_1} \right) + \frac{1}{h^3} \frac{\partial h}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \right]$$

$$+ 2\mu \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left[-\frac{1}{h} \frac{\partial}{\partial x_1} \left(\frac{1}{h} \frac{\partial \chi}{\partial x_2} \right) + \frac{1}{h^3} \frac{\partial h}{\partial x_2} \frac{\partial \chi}{\partial x_1} \right]$$

$$+ 2\mu \left[\frac{1}{h} \frac{\partial}{\partial x_1} \left(\frac{1}{h} \frac{\partial^2 \psi}{\partial x_1 \partial x_3} \right) + \frac{1}{h^3} \frac{\partial h}{\partial x_2} \frac{\partial^2 \psi}{\partial x_2 \partial x_3} \right] \qquad (52)$$

$$\sigma_{12} = \mu \left[\frac{\partial}{\partial x_1} \left(\frac{1}{h^2} \frac{\partial \varphi}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left(\frac{1}{h^2} \frac{\partial \varphi}{\partial x_1} \right) \right]$$

$$+ \mu \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left[\frac{\partial}{\partial x_1} \left(\frac{1}{h^2} \frac{\partial \chi}{\partial x_1} \right) - \frac{\partial}{\partial x_2} \left(\frac{1}{h^2} \frac{\partial \chi}{\partial x_2} \right) \right]$$

$$+ \mu \left[\frac{\partial}{\partial x_1} \left(\frac{1}{h^2} \frac{\partial^2 \psi}{\partial x_2 \partial x_3} \right) + \frac{\partial}{\partial x_2} \left(\frac{1}{h^2} \frac{\partial^2 \psi}{\partial x_1 \partial x_3} \right) \right] \qquad (53)$$

$$\sigma_{13} = \mu \left[\frac{2}{h} \frac{\partial^2 \varphi}{\partial x_1 \partial x_3} \right]$$

$$+ \mu \left[-\frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{1}{h} \frac{\partial^2 \chi}{\partial x_2 \partial x_3} \right) \right]$$

$$+ \mu \left[\frac{2}{h} \frac{\partial^3 \psi}{\partial x_1 \partial x_3^2} - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left(\frac{1}{h} \frac{\partial \psi}{\partial x_1} \right) \right] \qquad (54)$$

Seeking solutions to (42) representing waves travelling along the x_3-axis (the rod axis), and which are separable in x_1 and x_2, φ is of the form

$$\varphi\,(x_1, x_2, x_3, t) = \Phi_1(x_1)\Phi_2(x_2)e^{i(kx_3 - wt)}$$

Substituting this into (42), and using (43), gives

$$\frac{d^2\Phi_2}{dx_2^2} + (\eta - 2q_1 \cos 2x_2)\, \Phi_2 = 0 \tag{55}$$

$$\frac{d^2\Phi_1}{dx_1^2} - (\eta - 2q_1 \cosh 2x_1)\, \Phi_1 = 0 \tag{56}$$

where

$$q_1 = \frac{f^2}{4}\, p_1^2 \tag{57}$$

p_1^2 is as given by (31), and η is a separation constant. Equation (55) is Mathieu's equation and (56) is Mathieu's modified equation. Periodic solutions to these equations exist, provided η takes on certain values (functions of q_1), the so-called characteristic numbers. In general (see McLachlan [21]).

$$\Phi_2(x_2) = \left(\begin{array}{ll} ce_m\ (x_2, q_1) & \eta = a_m \\ se_m\ (x_2, q_1) & \eta = b_m \end{array}\right) \tag{58}$$

$$\Phi_1(x_1) = \left(\begin{array}{ll} Ce_m\ (x_1, q_1) & \eta = a_m \\ Se_m\ (x_1, q_1) & \eta = b_m \end{array}\right) \tag{59}$$

Here m is an integer, for single-valuedness, ce_m is the cosine-elliptic function, characteristic number a_m, se_m is the sine-elliptic function, characteristic number b_m, Ce_m is the modified cosine elliptic function characteristic number a_m, and Se_m is the modified sine-elliptic function, characteristic number b_m. For example

$$ce_1\ (x_2,q_1) = \text{cox}\ x_2 - \frac{1}{8}\,q_1 \cos 3x_2 + \frac{1}{64}\,q_1^2\ (-\cos 3x_2 + \frac{1}{3}\cos 5x_2) + \ldots$$

where
$$a_1 - 1 + q_1 - \frac{1}{8}\,q_1^2 - \frac{1}{64}\,q_1^3 - \cdots$$

The second linearly independent solution to (56), the form of which depends on whether $\eta = a_m$, or b_m, is rejected because it and its derivative which arise in the expressions for the displacements are not continuous across the interfocal line. In fact these solutions can be expressed as infinite series of Y function (second solutions of the Bessel's equation) and so have the same types of singularities as the Y functions. The second solution to (55) is rejected because it is not periodic in x_2. Other solutions to Mathieu's equation exist (the Floquet solutions; see Abramowitz and Stegun [24]) but they are rejected because of possible instabilities.

Similarly, substituting expression of the form

$$\psi(x_1,x_2,x_3,t) = \psi_1(x_1)\psi_2(x_2)\ e^{i(kx_3-wt)}$$

$$X(x_1,x_2,x_3,t) = X_1(x_1)X_2(x_2)\ e^{i(kx_3-wt)}$$

into (18) and (19), gives

$$\psi_2(x_2) = \begin{pmatrix} ce_m\ (x_2,q_2),\ \text{characteristic number}\ \bar{a}_m \\ \\ se_m\ (x_2,q_2),\ \text{characteristic number}\ \bar{b}_m \end{pmatrix} \qquad (60)$$

$$\psi_1(x_1) = \begin{pmatrix} Ce_m\ (x_1,q_2),\ \text{characteristic number}\ \bar{a}_m \\ \\ Se_m\ (x_1,q_2),\ \text{characteristic number}\ \bar{b}_m \end{pmatrix} \qquad (61)$$

$$X_2(x_2) = \begin{pmatrix} ce_m\ (x_2, q_2), \text{ characteristic number } \bar{a}_m \\ \\ se_m\ (x_2, q_2), \text{ characteristic number } \bar{b}_m \end{pmatrix} \tag{62}$$

$$X_1(x_1) = \begin{pmatrix} Ce_m\ (x_1, q_2), \text{ characteristic number } \bar{a}_m \\ \\ Se_m\ (x_1, q_2), \text{ characteristic number } \bar{b}_m \end{pmatrix} \tag{63}$$

where
$$q_2 = \frac{f^2 p_2^2}{4} \tag{64}$$

p_2^2 is given by (32), and a bar denotes the quantity is a function of q_2, as opposed to q_1 (a notation used throughout the remainder of the text).

The Mathieu functions in the above expressions have the following representation formulas (see McLachlan [21], § 2.17):

$$ce_{2m}(\alpha, q) = \sum_{r=0}^{\infty} A_{2r}^{(2m)} \cos 2r\alpha \quad (\text{characteristic number } a_{2m}) \tag{65}$$

$$ce_{2m+1}(\alpha, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)} \cos(2r+1)\alpha \quad (\text{characteristic number } a_{2m+1}) \tag{66}$$

$$se_{2m+1}(\alpha, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \sin(2r+1)\alpha \quad (\text{characteristic number } b_{2m+1}) \tag{67}$$

$$se_{2m+2}(\alpha, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \sin(2r+2)\alpha \quad (\text{characteristic number } b_{2m+2}) \tag{68}$$

Here the A's and B's are known functions of q. For instance

$$A_1^{(1)} = 1\ ;\ A_3^{(1)} = -\frac{1}{8}\, q\left(1 + \frac{1}{8}\, q + \frac{1}{192}\, q^2 + \cdots\right), \text{ etc.} \tag{69}$$

In any product pair solutions, Ce_m or Se_m is constant on any confocal ellipse. Hence the symmetry is governed by ce_m or se_m and it can be seen from (65) through (68) that the following four cases arise: (i) $Ce_{2m}(x_1, q_1)\, ce_{2m}(x_2, q_1)$ is symmetrical about both the major and minor axes, (ii) $Ce_{2m+1}(x_1, q_1)\, ce_{2m+1}(x_2, q_1)$ is symmetrical about the major axis, but antisymmetrical about the minor axis, (iii) $Se_{2m+1}(x_1, q_1)\, se_{2m+1}(x_2, q_1)$ is antisymmetrical about the major axis, but symmetrical about the minor axis, (iv) $Se_{2m+2}(x_1, q_1)\, se_{2m+2}(x_2, q_1)$ is antisymmetrical about the major and minor axes. It should be noted that these are the only combinations which arise, since solutions of (55) and (56) corresponding to the same characteristic number must be chosen.

The product solutions arising from (58) through (63), and linear combination of them, are solutions of the potential equations of motion. In common with circular rod problems, not all of these solutions are necessary to describe a specific problem and only certain combinations will be considered. Two specific problems will now be considered, namely, (i) the infinite elliptical rod whose surface is in welded contact with a rigid medium, i.e., all surface displacements are zero (ii) the elliptical rod whose surface is stress-free. It is found that four cases, or modes of motion, arise in both these problems and these will now be discussed.

(a) The solutions of the potential equations of motion are taken as

$$\varphi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} D_{2m}\, Ce_{2m}\,(x_1,q_1)\, ce_{2m}\,(x_2,q_1)\, e^{i(kx_3-wt)} \tag{70}$$

$$\psi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} F_{2m}\, Ce_{2m}\,(x_1,q_2)\, ce_{2m}\,(x_2,q_2)\, e^{i(kx_3-wt)} \tag{71}$$

$$\chi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} H_{2m+2}\, Se_{2m+2}\,(x_1,q_2)\, se_{2m+2}\,(x_2,q_2)\, e^{i(kx_3-wt)} \tag{72}$$

where D_{2m}, F_{2m}, and H_{2m+2} are constants. Inspection of the displacements, as given by (45), (46), and (47), and noting the symmetry properties described above, shows that with this choice of potentials, u_1 and u_3 are symmetric with respect to both major and minor axes, and u_2 is antisymmetric about both axes. Thus the mode of motion described by (70), (71), and (72) corresponds to compressional waves in a circular rod and it may be generated by the uniform normal loading of the end of a semi-infinite elliptical rod. It should be noted that the situation here is markedly different from that for the circular rod. In the circular rod case no infinite superposition is necessary, but here approaches taking only one term of (70) through (72) do not appear to be fruitful. The choice of the sine-elliptic function in (72) is perhaps best understood by recalling that in the circular rod case (for flexure) cos θ appeared in the φ and χ-expressions, whereas sin θ appeared in the ψ-expression.

(b) The solutions of the potential equations of motion are taken as

$$\varphi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} D_{2m+1}\, Ce_{2m+1}\,(x_1,q_1)\, ce_{2m+1}\,(x_2,q_1)\, e^{i(kx_3-wt)} \tag{73}$$

$$\psi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} F_{2m+1}\, Ce_{2m+1}\,(x_1,q_2)\, ce_{2m+1}\,(x_2,q_2)\, e^{i(kx_3-wt)} \tag{74}$$

$$\chi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} H_{2m+1}\, Se_{2m+1}\,(x_1,q_2)\, se_{2m+1}\,(x_2,q_2)\, e^{i(kx_3-wt)} \tag{75}$$

where D_{2m+1}, F_{2m+1}, and H_{2m+1}, are constants. In this case it is found that u_1 and u_3 are symmetrical about the major axis, but antisymmetrical about the minor axis, whereas u_2 is antisymmetrical about the major axis, but symmetrical about the minor axis. The mode of motion in this case corresponds to a flexural mode in the circular rod and may be generated by end surface loads acting parallel to, and symmetrical about, the major axis of the ellipse.

(c) The solutions of the potential equations of motion are taken as

$$\varphi(x_1,x_2,x_3,t) = \sum_{m=o}^{\infty} I_{2m+1}Se_{2m+1}(x_1,q_1)se_{2m+1}(x_2,q_1)\ e^{i(kx_3-wt)} \tag{76}$$

$$\psi(x_1,x_2,x_3,t) = \sum_{m=o}^{\infty} T_{2m+1}Se_{2m+1}(x_1,q_2)se_{2m+1}(x_2,q_2)\ e^{i(kx_3-wt)} \tag{77}$$

$$\chi(x_1,x_2,x_3,t) = \sum_{m=o}^{\infty} L_{2m+1}Ce_{2m+1}(x_1,q_2)ce_{2m+1}(x_2,q_2)\ e^{i(kx_3-wt)} \tag{78}$$

where I_{2m+1}, T_{2m+1}, and L_{2m+1}, are constants. In this case it is found that u_1 and u_3 are antisymmetrical about the major axis, but symmetrical about the minor axis, whereas u_2 is symmetrical about the major axis, but antisymmetrical about the minor axis. This mode of motion is also of the flexural type, but now the external generating loads act parallel to, and symmetrical about, the minor axis of the ellipse. For external loads acting in an arbitrary direction, combinations of (b) and (c) should be taken.

(d) The solutions of the potential equations of motion are taken as:

$$\varphi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} I_{2m+2}Se_{2m+2}(x_1,q_1)se_{2m+2}(x_2,q_1)\ e^{i(kx_3-wt)} \tag{79}$$

$$\psi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} T_{2m+2}Se_{2m+2}(x_1,q_2)se_{2m+2}(x_2,q_2)e^{i(kx_3-wt)} \tag{80}$$

$$X(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty} L_{2m}Ce_{2m}(x_1,q_2)ce_{2m}(x_2,q_2)\ e^{i(kx_3-wt)} \tag{81}$$

where I_{2m+2}, T_{2m+2}, and L_{2m}, are constants. In this case it is found that u_1 and u_3 are antisymmetrical about the major and minor axes, whereas u_2 is symmetrical about the major axis, but antisymmetrical about the minor axis. This mode of motion is the analogue of torsional waves in a circular rod. The particular physical problems at hand will now be discussed.

(i) Frequency equation for "flexural" waves in an elliptical rod, whose surface is in welded contact with a rigid medium.

Though this problem is of interest in its own right, its major usage here will be to illustrate the solution procedure. Substituting (66) and (67) into (73), (74) and (75), gives, as an alternative form for the potentials,

$$\varphi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} D_{2m+1}A_{2r+1}^{(2m+1)}Ce_{2m+1}(x_1,q_1)\cos(2r+1)x_2e^{i(kx_3-wt)} \tag{82}$$

$$\psi(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} F_{2m+1}\ \overline{A}_{2r+1}^{(2m+1)}Ce_{2m+1}(x_1,q_2)\cos(2r+1)x_2e^{i(kx_3-wt)} \tag{83}$$

$$X(x_1,x_2,x_3,t) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} H_{2m+1}\overline{B}_{2r+1}^{(2m+1)}Se_{2m+1}(x_1,q_2)\sin(2r+1)x_2e^{i(kx_3-wt)} \tag{84}$$

Substituting (82), (83), and (84) into (45), (46), and (47), gives for the displacements

$$u_1 = \frac{1}{h}\sum_{m=o}^{\infty}\sum_{r=o}^{\infty}\left\{ D_{2m+1}A_{2r+1}^{(2m+1)}\frac{d}{dx_1}Ce_{2m+1}(x_1,q_1)\right.$$

$$+ F_{2m+1}\, ik\, \bar{A}_{2r+1}^{(2m+1)}\frac{d}{dx_1}Ce_{2m+1}(x_1,q_2) \tag{85}$$

$$\left. + H_{2m+1}\frac{\omega^2}{c_s^2}(2r+1)\bar{B}_{2r+1}^{(2m+1)}Se_{2m+1}(x_1,q_2)\right\}\cos(2r+1)x_2 e^{i(kx_3-wt)}$$

$$u_2 = -\frac{1}{h}\sum_{m=o}^{\infty}\sum_{r=o}^{\infty}\left\{ D_{2m+1}(2r+1)A_{2r+1}^{(2m+1)}Ce_{2m+1}(x_1,q_1)\right.$$

$$+ F_{2m+1}\, ik\,(2r+1)\bar{A}_{2r+1}^{(2m+1)}Ce_{2m+1}(x_1,q_2) \tag{86}$$

$$\left. + H_{2m+1}\frac{\omega^2}{c_s^2}\bar{B}_{2r+1}^{(2m+1)}\frac{d}{dx_1}Se_{2m+1}(x_1,q_2)\right\}\sin(2r+1)x_2 e^{i(kx_3-wt)}$$

$$u_3 = \sum_{m=o}^{\infty}\sum_{r=o}^{\infty}\left\{ ik\, D_{2m+1}A_{2r+1}^{(2m+1)}Ce_{2m+1}(x_1,q_1)\right.$$

$$\left. + p_2^2 F_{2m+1}\bar{A}_{2r+1}^{(2m+1)}Ce_{2m+1}(x_1,q_2)\right\}\cos(2r+1)x_2 e^{i(kx_3-wt)} \tag{87}$$

Let the boundary of the ellipse be specified by $x_1 = \xi$ (constant). Then the boundary conditions are

$$u_1 = u_2 = u_3 = o, \qquad x_1 = \xi .$$

Applying these to (85), (86), and (87), gives, on interchanging the order of summation,

$$\sum_{r=o}^{\infty}\sum_{m=o}^{\infty}\left\{D_{2m+1}A_{2r+1}^{(2m+1)}\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_1)\right]_{x_1=\xi}\right.$$
$$+\frac{\omega^2}{c_s^2}(2r+1)\ H_{2m+1}\ \overline{B}_{2r+1}^{(2m+1)}\ Se_{2m+1}(\xi,q_2)$$
$$\left.+\ ik\ F_{2m+1}\ \overline{A}_{2r+1}^{(2m+1)}\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_2)\right]_{x_1=\xi}\right\}\cos(2r+1)\ x_2=0 \qquad (88)$$

$$\sum_{r=o}^{\infty}\sum_{m=o}^{\infty}\left\{(2r+1)\ D_{2m+1}\ A_{2r+1}^{(2m+1)}\ Ce_{2m+1}(\xi,q_1)\right.$$
$$+\ ik\ (2r+1)\ F_{2m+1}\ \overline{A}_{2r+1}^{(2m+1)}\ Ce_{2m+1}(\xi,q_2)$$
$$\left.+\frac{\omega^2}{c_s^2}\ H_{2m+1}\ \overline{B}_{2r+1}^{(2m+1)}\left[\frac{d}{dx_1}\ Se_{2m+1}(x_1,q_2)\right]_{x_1=\xi}\right\}\sin(2r+1)\ x_2=0 \qquad (89)$$

$$\sum_{r=o}^{\infty}\sum_{m=o}^{\infty}\left\{ik\ D_{2m+1}\ A_{2r+1}^{(2m+1)}\ Ce_{2m+1}(\xi,q_1)\right.$$
$$\left.+\ p_2^2\ F_{2m+1}\ \overline{A}_{2r+1}^{(2m+1)}\ Ce_{2m+1}(\xi,q_2)\right\}\cos(2r+1)\ x_2=0 \qquad (90)$$

The satisfaction of these equations requires that, using the linear independence of the sets of functions $\cos(2r+1)x_2$, $\sin(2r+1)x_2$, $r=0,\ 1,\ \cdots$:

$$D_1\left\{A_1^{(1)}\left[\frac{d}{dx_1}\ Ce_1(x_1,q_1)\right]_{x_1=\xi}\right\}+H_1\left\{\frac{\omega^2}{c_s^2}\ \overline{B}_1^{(1)}\ Se_1(\xi,q_2)\right\}$$
$$+\ F_1\left\{ik\ \overline{A}_1^{(1)}\left[\frac{d}{dx_1}\ Ce_1(x_1,q_2)\right]_{x_1=\xi}\right\}$$
$$+\ D_3\left\{A_1^{(3)}\left[\frac{d}{dx_1}\ Ce_3(x_1,q_1)\right]_{x_1=\xi}\right\}+H_3\left\{\frac{\omega^2}{c_s^2}\ \overline{B}_1^{(3)}\ Se_3(\xi,q_2)\right\}$$
$$+\ F_3\left\{ik\ \overline{A}_1^{(3)}\left[\frac{d}{dx_1}\ Ce_3(x_1,q_2)\right]_{x_1=\xi}\right\}+\ \cdots\ =0 \qquad (91)$$

$$D_1 \left\{ A_1^{(1)} \; Ce_1(\xi, q_1) \right\} + H_1 \left\{ \frac{\omega^2}{c_s^2} \overline{B}_1^{(1)} \left[\frac{d}{dx_1} \; Se_1(x_1, q_2) \right]_{x_1=\xi} \right\}$$
$$+ F_1 \left\{ ik \; \overline{A}_1^{(1)} \; Ce_1(\xi, q_2) \right\} + D_3 \left\{ A_1^{(3)} \; Ce_3(\xi, q_1) \right\}$$
$$+ H_3 \left\{ \frac{\omega^2}{c_s^2} \overline{B}_1^{(3)} \left[\frac{d}{dx_1} \; Se_3(x_1, q_2) \right]_{x_1=\xi} \right\} + F_3 \left\{ ik \; \overline{A}_1^{(3)} Ce_3(\xi, q_2) \right\} + \cdots = 0 \tag{92}$$

$$D_1 \left\{ ik \; A_1^{(1)} \; Ce_1(\xi, q_1) \right\} + H_1 \left\{ 0 \right\} + F_1 \left\{ p_2^2 \; \overline{A}_1^{(1)} \; Ce_1(\xi, q_2) \right\}$$
$$+ D_3 \left\{ ik \; A_1^{(3)} \; Ce_3(\xi, q_1) \right\} + H_3 \left\{ 0 \right\} + F_3 \left\{ p_2^2 \; \overline{A}_1^{(3)} Ce_3(\xi, q_2) \right\} + \cdots = 0 \tag{93}$$

$$D_1 \left\{ A_3^{(1)} \left[\frac{d}{dx_1} \; Ce_1(x_1, q_1) \right]_{x_1=\xi} \right\} + H_1 \left\{ 3 \; \frac{\omega^2}{c_s^2} \; \overline{B}_3^{(1)} \; Se_1(\xi, q_2) \right\}$$
$$+ F_1 \left\{ ik \; \overline{A}_3^{(1)} \left[\frac{d}{dx_1} \; Ce_1(x_1, q_2) \right]_{x_1=\xi} \right\} + D_3 \left\{ \overline{A}_3^{(3)} \left[\frac{d}{dx_1} \; Ce_3(x_1, q_1) \right]_{x_1=\xi} \right\}$$
$$+ H_3 \left\{ 3 \; \frac{\omega^2}{c_s^2} \; \overline{B}_3^{(3)} \; Se_3(\xi, q_2) \right\} + F_3 \left\{ ik \; \overline{A}_3^{(3)} \left[\frac{d}{dx_1} \; Ce_3(x_1, q_2) \right]_{x_1=\xi} \right\} + \cdots = 0 \tag{94}$$

$$D_1 \left\{ 3A_3^{(1)} \; Ce_1(\xi, q_1) \right\} + H_1 \left\{ \frac{\omega^2}{c_s^2} \overline{B}_3^{(1)} \left[\frac{d}{dx_1} \; Se_1(x_1, q_2) \right]_{x_1=\xi} \right\}$$
$$+ F_1 \left\{ 3ik \; \overline{A}_3^{(1)} \; Ce_1(\xi, q_2) \right\} + D_3 \left\{ 3A_3^{(3)} \; Ce_3(\xi, q_1) \right\}$$
$$+ H_3 \left\{ \frac{\omega^2}{c_s^2} \overline{B}_3^{(3)} \left[\frac{d}{dx_1} \; Se_3(x_1, q_2) \right]_{x_1=\xi} \right\}$$
$$+ F_3 \left\{ 3ik \; \overline{A}_3^{(3)} \; Ce_3(\xi, q_2) \right\} + \cdots = 0 \tag{95}$$

$$D_1 \left\{ ik \; A_3^{(1)} \; Ce_1(\xi, q_1) \right\} + H_1 \left\{ 0 \right\} + F_1 \left\{ p_2^2 \; \overline{A}_3^{(1)} Ce_1(\xi, q_2) \right\}$$
$$+ D_3 \left\{ ik \; A_3^{(3)} \; Ce_3(\xi, q_1) \right\} + H_3 \left\{ 0 \right\} + F_3 \left\{ p_2^2 \; \overline{A}_3^{(3)} \; Ce_3(\xi, q_2) \right\} + \cdots = 0 \tag{96}$$

Thus it is seen that these are an infinite number of equations to determine the infinite number of unknowns D_1, H_1, F_1, D_3, H_3, F_3, $\cdots$, and for their satisfaction the determinant (infinite) of the coefficients must equal zero, and so the frequency equation is of the form:

$$\begin{vmatrix} \Delta_{11} & \Delta_{13} & \cdots \\ \Delta_{31} & \Delta_{33} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} = 0 \tag{97}$$

where

$$\Delta_{11} = \begin{bmatrix} A_1^{(1)}\left[\frac{d}{dx_1}Ce_1(x_1,q_1)\right]_{x_1=\xi} & \frac{\omega^2}{c_s^2}\bar{B}_1^{(1)}Se_1(\xi,q_2) & ik\bar{A}_1^{(1)}\left[\frac{d}{dx_1}Ce_1(x_1,q_2)\right]_{x_1=\xi} \\ A_1^{(1)}\;Ce_1(\xi,q_1) & \frac{\omega^2}{c_s^2}\;\bar{B}_1^{(1)}\left[\frac{d}{dx_1}\;Se_1(x_1,q_2)\right]_{x_1=\xi} & ik\;\bar{A}_1^{(1)}Ce_1(\xi,q_2) \\ ik\;A_1^{(1)}Ce_1(\xi,q_1) & 0 & p_2^2\;\bar{A}^{(1)}Ce_1(\xi,q_2) \end{bmatrix} \tag{98}$$

$$\Delta_{13} = \begin{bmatrix} A_1^{(3)}\left[\frac{d}{dx_1}Ce_3(x_1,q_1)\right]_{x_1=\xi} & \frac{\omega^2}{c_s^2}\bar{B}_1^{(3)}Se_3(\xi,q_2) & ik\bar{A}_1^{(3)}\left[\frac{d}{dx_1}Ce_3(x_1,q_2)\right]_{x_1=\xi} \\ A_1^{(3)}\;Ce_3(\xi,q_1) & \frac{\omega^2}{c_s^2}\;\bar{B}_1^{(3)}\left[\frac{d}{dx_1}\;Se_3(x_1,q_2)\right]_{x_1=\xi} & ik\bar{A}_1^{(3)}Ce_3(\xi,q_2) \\ ik\;A_1^{(3)}Ce_3(\xi,q_1) & 0 & p_2^2\bar{A}_1^{(3)}Ce_3(\xi,q_2) \end{bmatrix} \tag{99}$$

$$\Delta_{31} = \begin{bmatrix} A_3^{(1)}\left[\frac{d}{dx_1}Ce_1(x_1,q_1)\right]_{x_1=\xi} & 3\frac{\omega^2}{c_s^2}\bar{B}_3^{(1)}Se_1(\xi,q_2) & ik\bar{A}_3^{(1)}\left[\frac{d}{dx_1}Ce_1(x_1,q_2)\right]_{x_1=\xi} \\ 3A_3^{(1)}Ce_1(\xi,q_1) & \frac{\omega^2}{c_s^2}\bar{B}_3^{(1)}\left[\frac{d}{dx_1}Se_1(x_1,q_2)\right]_{x_1=\xi} & 3ik\bar{A}_3^{(1)}Ce_1(\xi,q_2) \\ ik\,A_3^{(1)}Ce_1(\xi,q_1) & 0 & p_2^2\,\bar{A}_3^{(1)}Ce_1(\xi,q_2) \end{bmatrix} \tag{100}$$

$$\Delta_{33} = \begin{bmatrix} A_3^{(3)}\left[\frac{d}{dx_1}Ce_3(x_1,q_1)\right]_{x_1=\xi} & 3\frac{\omega^2}{c_s^2}\bar{B}_3^{(3)}Se_3(\xi,q_2) & ik\bar{A}_3^{(3)}\left[\frac{d}{dx_1}Ce_3(x_1,q_2)\right]_{x_1=\xi} \\ 3A_3^{(3)}Ce_3(\xi,q_1) & \frac{\omega^2}{c_s^2}\bar{B}_3^{(3)}\left[\frac{d}{dx_1}Se_3(x_1,q_2)\right]_{x_1=\xi} & 3ik\bar{A}_3^{(3)}Ce_3(\xi,q_2) \\ ik\,A_3^{(3)}Ce_3(\xi,q_1) & 0 & p_2^2\,\bar{A}_3^{(3)}Ce_3(\xi,q_2) \end{bmatrix} \tag{101}$$

It is interesting to see what happens to the frequency equation (97) in the event that the elliptical rod becomes circular. This is achieved by setting the eccentricity equal to zero, or equivalently, from (41), (57), and (64), $q_1 = q_2 = 0$. Inspection of the coefficients $A_{2r+1}^{(2m+1)}$, $\bar{A}_{2r+1}^{(2m+1)}$, and $\bar{B}_{2r+1}^{(2m+1)}$, shows that in this case they reduce to 1, for $r = m$, and to 0 for $r \neq m$ (see, for instance, (69)). Hence (97) reduces to a diagonal determinant and the frequency equation is of the form

$$\Delta_{11} \cdot \Delta_{33} \cdots = 0$$

but (see McLachlan [21], pp. 368-369).

$$\left.\begin{aligned}
&Ce_{2m+1}(x_1,q_j) \xrightarrow[e\to o]{} k_{2m+1}\, J_{2m+1}(p_j a) \\
&Se_{2m+1}(x_1,q_j) \xrightarrow[e\to o]{} k'_{2m+1}\, J_{2m+1}(p_j a) \\
&\frac{d}{dx_1}\, Ce_{2m+1}(x_1,q_j) \xrightarrow[e\to o]{} k_{2m+1}\, a \left[\frac{d}{dr} J_{2m+1}(p_j r)\right]_{r=a} \\
&\frac{d}{dx_1}\, Se_{2m+1}(x_1,q_j) \xrightarrow[e\to o]{} k'_{2m+1}\, a \left[\frac{d}{dr} J_{2m+1}(p_j r)\right]_{r=a}
\end{aligned}\right\} \; j=1,2 \qquad (102)$$

where k_{2m+1} and k'_{2m+1} are constants,* for fixed m, and a is now the circular rod radius. Hence (98) reduces to:

$$\Delta_{11} = \frac{\omega^2}{c_s^2}\, k_1^2\, k'_1 \begin{vmatrix} a\left[\frac{d}{dr} J_1(p_1 r)\right]_{r=a} & J_1(p_2 a) & ika\left[\frac{d}{dr} J_1(p_2 r)\right]_{r=a} \\ J_1(p_1 a) & a\left[\frac{d}{dr} J_1(p_2 r)\right]_{r=a} & ik\, J_1(p_2 a) \\ ik\, J_1(p_1 a) & 0 & p_2^2\, J_1(p_2 a) \end{vmatrix}$$

This last expression set equal to zero can readily be shown to be the frequency equation describing the propagation of flexural waves in an infinite circular rod, of radius a, when the surface displacements are zero. Similarly it can be shown that Δ_{33} set equal to zero reduces to the frequency equation for the propagation of a higher circumferential type mode (i.e., (28), (29), and (30), with $n = m = \ell = 3$) in an infinite circular rod when the surface displacements are zero.

---

*In the form given by McLachlan these constants are ratios of functions of q. Their value as q goes to zero is of no interest here, since they cancel in the final result.

(ii) Frequency equation for "flexural" waves in an elliptical rod whose surface is stress-free.

The stress-potential relationships, given by (52), (53), and (54), may be written, on substituting for the derivatives of the metric coefficients and doing some rearranging:

$$
\begin{aligned}
\sigma_{11} = \frac{1}{h^4}\Bigg\{ & \frac{\lambda}{c_d^2} h^4 \frac{\partial^2\varphi}{\partial t^2} + \mu\Bigg[2h^2 \frac{\partial^2\varphi}{\partial x_1^2} - f^2 \sinh 2x_1 \frac{\partial\varphi}{\partial x_1} + f^2 \sin 2x_2 \frac{\partial\varphi}{\partial x_2} \\
& + 2h^2 \frac{\partial^3\psi}{\partial x_1^2\partial x_3} - f^2 \sinh 2x_1 \frac{\partial^2\psi}{\partial x_1\partial x_3} + f^2 \sin 2x_2 \frac{\partial^2\psi}{\partial x_2\partial x_3} \Bigg] \\
& - \frac{\mu}{c_s^2} \frac{\partial^2}{\partial t^2}\Bigg[2h^2 \frac{\partial^2\chi}{\partial x_1\partial x_2} - f^2 \sinh 2x_1 \frac{\partial\chi}{\partial x_2} - f^2 \sin 2x_2 \frac{\partial\chi}{\partial x_1}\Bigg]\Bigg\}
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
\sigma_{12} = \frac{\mu}{h^4}\Bigg\{ & 2h^2 \frac{\partial^2\varphi}{\partial x_1\partial x_2} - f^2 \sinh 2x_1 \frac{\partial\varphi}{\partial x_2} - f^2 \sin 2x_2 \frac{\partial\varphi}{\partial x_1} + 2h^2 \frac{\partial^3\psi}{\partial x_1\partial x_2\partial x_3} \\
& - f^2 \sinh 2x_1 \frac{\partial^2\psi}{\partial x_2\partial x_3} - f^2 \sin 2x_2 \frac{\partial^2\psi}{\partial x_1\partial x_3} + \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2}\Bigg[h^2\left(\frac{\partial^2\chi}{\partial x_1^2} - \frac{\partial^2\chi}{\partial x_2^2}\right) \\
& + f^2 \sin 2x_2 \frac{\partial\chi}{\partial x_2} - f^2 \sinh 2x_1 \frac{\partial\chi}{\partial x_1}\Bigg]\Bigg\}
\end{aligned} \tag{104}
$$

$$
\sigma_{13} = \frac{\mu}{h}\left\{ 2 \frac{\partial^2\varphi}{\partial x_1\partial x_3} + \frac{\partial}{\partial x_1}\left(-\frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2} + 2\frac{\partial^2\psi}{\partial x_3^2}\right) - \frac{1}{c_s^2}\frac{\partial^4\chi}{\partial t^2\partial x_2\partial x_3}\right\} \tag{105}
$$

Substituting (73), (74), and (75), into (103), (104), and (105), gives, on using (66) and (67),

$$
\begin{aligned}
h^4\sigma_{11} = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} D_{2m+1}\Bigg\{ & -\frac{\omega^2\lambda}{c_d^2} h^4 \, Ce_{2m+1}(x_1,q_1) + 2\mu h^2\left[\frac{d^2}{dx_1^2} Ce_{2m+1}(x_1,q_1)\right] \\
& - \mu f^2 \sinh 2x_1\left[\frac{d}{dx_1} Ce_{2m+1}(x_1,q_1)\right]\Bigg\} A_{2r+1}^{(2m+1)} \cos(2r+1)\, x_2 \, e^{i(kx_3 - wt)}
\end{aligned}
$$

$$-\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} D_{2m+1}\ \mu\ f^2\ Ce_{2m+1}(x_1,q_1)\ (2r+1)A_{2r+1}^{(2m+1)} \sin 2x_2 \sin(2r+1)x_2 e^{i(kx_3-wt)}$$

$$+\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} F_{2m+1}\left\{2\mu h^2 ik\left[\frac{d^2}{dx_1^2}\ Ce_{2m+1}(x_1,q_2)\right] - \mu f^2 ik\ \sinh 2x_1\left[\frac{d}{dx_1}Ce_{2m+1}(x_1,q_2)\right]\right\}$$

$$\times\ \bar{A}_{2r+1}^{(2m+1)} \cos(2r+1)x_2\ e^{i(kx_3-wt)}$$

$$-\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} F_{2m+1}\ \mu\ f^2 ik Ce_{2m+1}(x_1,q_2)(2r+1)\bar{A}_{2r+1}^{(2m+1)} \sin 2x_2 \sin(2r+1)x_2 e^{i(kx_3-wt)}$$

$$+\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} H_{2m+1}\left\{\frac{\mu\omega^2}{c_s^2}\ 2h^2\left[\frac{d}{dx_1}\ Se_{2m+1}(x_1,q_2)\right]\right.$$

$$\left. -\frac{\mu\omega^2}{c_s^2}\ f^2\ \sinh 2x_1\ Se_{2m+1}(x_1,q_2)\right\}(2r+1)\bar{B}_{2r+1}^{(2m+1)} \cos(2r+1)x_2\ c^{i(kx_3-wt)}$$

$$-\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} H_{2m+1}\ \frac{\mu\omega^2 f^2}{c_s^2}\left[\frac{d}{dx_1}\ Se_{2m+1}(x_1,q_2)\right]\bar{B}_{2r+1}^{(2m+1)} \sin 2x_2 \sin(2r+1)x_2 e^{i(kx_3-wt)}$$

(106)

$$\frac{h^4}{\mu}\ \sigma_{12} = -\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} D_{2m+1}\left\{2h^2\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_1)\right]\right.$$

$$\left. -\ f^2\ \sinh\ 2x_1\ Ce_{2m+1}(x_1,q_1)\right\}\ (2r+1)A_{2r+1}^{(2m+1)} \sin(2r+1)x_2\ e^{i(kx_3-wt)}$$

$$-\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} D_{2m+1}\ f^2\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_1)\right]A_{2r+1}^{(2m+1)} \sin 2x_2 \cos(2r+1)x_2\ e^{i(kx_3-wt)}$$

$$-\sum_{m=o}^{\infty}\sum_{r=o}^{\infty} F_{2m+1}\left\{ik\ 2h^2\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_2)\right]\right.$$

$$\left. -\ f^2\ ik\ \sinh\ 2x_1\ Ce_{2m+1}(x_1,q_2)\right\}\ (2r+1)\bar{A}_{2r+1}^{(2m+1)}\ \sin(2r+1)\ x_2\ e^{i(kx_3-wt)}$$

$$-\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} F_{2m+1}\ f^2\ ik\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_2)\right]\overline{A}_{2r+1}^{(2m+1)}\sin 2x_2 \cos(2r+1)x_2 e^{i(kx_3-wt)}$$

$$-\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} H_{2m+1}\ \frac{\omega^2}{c_s^2}\left\{h^2\left[\frac{d^2}{dx_1^2}\ Se_{2m+1}(x_1,q_2)\right]+h^2\ (2r+1)^2\ Se_{2m+1}\ (x_1,q_2)\right.$$

$$\left.-\ f^2\ \sinh\ 2x_1\left[\frac{d}{dx_1}\ Se_{2m+1}(x_1,q_2)\right]\right\}\overline{B}_{2r+1}^{(2m+1)}\ \sin\ (2r+1)\ x_2\ e^{i(kx_3-wt)}$$

$$-\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} H_{2m+1}\ \frac{\omega^2}{c_s^2}\ f^2\ Se_{2m+1}(x_1,q_2)(2r+1)\overline{B}_{2r+1}^{(2m+1)}\sin 2x_2\cos(2r+1)x_2 e^{i(kx_3-wt)} \tag{107}$$

$$\frac{h}{\mu}\ \sigma_{13} = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} D_{2m+1} 2ik\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_1)\right]\ A_{2r+1}^{(2m+1)}\cos(2r+1)x_2\ e^{i(kx_3-wt)}$$

$$+\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} F_{2m+1}\ \left(\frac{\omega^2}{c_s^2}-2k^2\right)\left[\frac{d}{dx_1}\ Ce_{2m+1}(x_1,q_2)\right]\overline{A}_{2r+1}^{(2m+1)}\cos(2r+1)x_2\ e^{i(kx_3-wt)}$$

$$+\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} H_{2m+1}\ \frac{\omega^2}{c_s^2}\ ik\ Se_{2m+1}(x_1,q_2)(2r+1)\overline{B}_{2r+1}^{(2m+1)}\cos(2r+1)x_2\ e^{i(kx_3-wt)} \tag{108}$$

It should be noticed that both sines and cosines of x_2, and products of sines and cosines of x_2, arise in (106) and (107). To obtain a uniform dependence in these functions, which, as the previous example shows, is necessary to obtain the solution, the following trigonometric identities will be employed, listed in the order of their occurrence:

$$\sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}\cos\ 2x_2\ \cos(2r+1)x_2 = \sum_{r=0}^{\infty}\quad U_{2r+1}^{(2m+1)}\ \cos\ (2r+1)\ x_2 \tag{109}$$

where

$$U_1^{(2m+1)} = \tfrac{1}{2}\left[A_1^{(2m+1)} + A_3^{(2m+1)}\right] \tag{110}$$

$$U_{2r+1}^{(2m+1)} = \tfrac{1}{2}\left[A_{2r-1}^{(2m+1)} + A_{2r+3}^{(2m+1)}\right], \; r = 1,2,\cdots \tag{111}$$

$$\sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)} \cos\, 4x_2{}^{*} \cos(2r+1)x_2 = \sum_{r=0}^{\infty} W_{2r+1}^{(2m+1)} \cos(2r+1)x_2 \tag{112}$$

where

$$W_1^{(2m+1)} = \tfrac{1}{2}\left[A_3^{(2m+1)} + A_5^{(2m+1)}\right] \tag{113}$$

$$W_3^{(2m+1)} = \tfrac{1}{2}\left[A_1^{(2m+1)} + A_7^{(2m+1)}\right] \tag{114}$$

$$W_{2r+1}^{(2m+1)} = \tfrac{1}{2}\left[A_{2r-3}^{(2m+1)} + A_{2r+5}^{(2m+1)}\right], \; r = 2,3,\cdots \tag{115}$$

$$\sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(2m+1)} \sin\, 2x_2 \, \sin(2r+1)x_2 = \sum_{r=0}^{\infty} \Gamma_{2r+1}^{(2m+1)} \cos(2r+1)x_2 \tag{116}$$

where

$$\Gamma_1^{(2m+1)} = \tfrac{1}{2}\left[A_1^{(2m+1)} + 3\, A_3^{(2m+1)}\right] \tag{117}$$

$$\Gamma_{2r+1}^{(2m+1)} = \tfrac{1}{2}\left[(2r+3) A_{2r+3}^{(2m+1)} - (2r-1)\, A_{2r-1}^{(2m+1)}\right], \; r = 1,2,\cdots \tag{118}$$

$$\sum_{r=0}^{\infty} (2r+1) \overline{B}_{2r+1}^{(2m+1)} \cos 2x_2 \cos(2r+1)x_2 = \sum_{r=0}^{\infty} \overline{\Lambda}_{2r+1}^{(2m+1)} \cos(2r+1)x_2 \tag{119}$$

where

$$\overline{\Lambda}_1^{(2m+1)} = \tfrac{1}{2}\left[\overline{B}_1^{(2m+1)} + 3\, \overline{B}_3^{(2m+1)}\right] \tag{120}$$

$$\overline{\Lambda}_{2r+1}^{(2m+1)} = \tfrac{1}{2}\left[(2r-1)\, \overline{B}_{2r-1}^{(2m+1)} + (2r+3)\, \overline{B}_{2r+3}^{(2m+1)}\right], \; r = 1,2,\cdots \tag{121}$$

$$\sum_{r=0}^{\infty} \overline{B}_{2r+1}^{(2m+1)} \sin\, 2x_2 \, \sin(2r+1)x_2 = \sum_{r=0}^{\infty} \overline{R}_{2r+1}^{(2m+1)} \cos(2r+1)x_2 \tag{122}$$

where

$$\overline{R}_1^{(2m+1)} = \tfrac{1}{2}\left[\overline{B}_1^{(2m+1)} + \overline{B}_3^{(2m+1)}\right] \tag{123}$$

*The $\cos\, 4x_2$ arises because of the h^4 term in (106).

$$\overline{R}_{2r+1}^{(2m+1)} = \frac{1}{2}\left[\overline{B}_{2r+3}^{(2m+1)} - \overline{B}_{2r-1}^{(2m+1)}\right], \quad r = 1,2, \cdots \tag{124}$$

$$\sum_{r=o}^{\infty}(2r+1)\, A_{2r+1}^{(2m+1)} \cos 2x_2 \, \sin(2r+1)x_2 = \sum_{r=o}^{\infty} \gamma_{2r+1}^{(2m+1)} \sin(2r+1)x_2 \tag{125}$$

where
$$\gamma_1^{(2m+1)} = \frac{1}{2}\left[3A_3^{(2m+1)} - A_1^{(2m+1)}\right] \tag{126}$$

$$\gamma_{2r+1}^{(2m+1)} = \frac{1}{2}\left[(2r-1)\, A_{2r-1}^{(2m+1)} + (2r+3)\, A_{2r+3}^{(2m+1)}\right], \quad r = 1,2, \cdots \tag{127}$$

$$\sum_{r=o}^{\infty} A_{2r+1}^{(2m+1)} \sin 2x_2 \, \cos(2r+1)x_2 = \sum_{r=o}^{\infty} O_{2r+1}^{(2m+1)} \sin(2r+1)x_2 \tag{128}$$

where
$$O_1^{(2m+1)} = \frac{1}{2}\left[A_1^{(2m+1)} - A_3^{(2m+1)}\right] \tag{129}$$

$$O_{2r+1}^{(2m+1)} = \frac{1}{2}\left[A_{2r-1}^{(2m+1)} - A_{2r+3}^{(2m+1)}\right], \quad r = 1,2, \cdots \tag{130}$$

$$\sum_{r=o}^{\infty}(2r+1)^2 \overline{B}_{2r+1}^{(2m+1)} \cos 2x_2 \, \sin(2r+1)x_2 = \sum_{r=o}^{\infty} \overline{\delta}_{2r+1}^{(2m+1)} \sin(2r+1)x_2 \tag{131}$$

where
$$\overline{\delta}_1^{(2m+1)} = \frac{1}{2}\left[9\, \overline{B}_3^{(2m+1)} - \overline{B}_1^{(2m+1)}\right] \tag{132}$$

$$\overline{\delta}_{2r+1}^{(2m+1)} = \frac{1}{2}\left[(2r-1)^2\, \overline{B}_{2r-1}^{(2m+1)} + (2r+3)^2\, \overline{B}_{2r+3}^{(2m+1)}\right], \quad r = 1,2, \cdots \tag{133}$$

$$\sum_{r=o}^{\infty} \overline{B}_{2r+1}^{(2m+1)} \cos 2x_2 \, \sin(2r+1)x_2 = \sum_{r=o}^{\infty} \overline{\epsilon}_{2r+1}^{(2m+1)} \sin(2r+1)\, x_2 \tag{134}$$

where
$$\overline{\epsilon}_1^{(2m+1)} = \frac{1}{2}\left[\overline{B}_3^{(2m+1)} - \overline{B}_1^{(2m+1)}\right] \tag{135}$$

$$\overline{\epsilon}_{2r+1}^{(2m+1)} = \frac{1}{2}\left[\overline{B}_{2r-1}^{(2m+1)} + \overline{B}_{2r+3}^{(2m+1)}\right], \quad r = 1,2, \cdots \tag{136}$$

$$\sum_{r=o}^{\infty}(2r+1)\ \bar{B}_{2r+1}^{(2m+1)}\ \sin 2x_2\ \cos(2r+1)x_2 = \sum_{r=o}^{\infty} \mathcal{J}_{2r+1}^{(2m+1)}\ \sin(2r+1)x_2 \qquad (137)$$

where
$$\mathcal{J}_{1}^{(2m+1)} = \tfrac{1}{2}\left[\ \bar{B}_{1}^{(2m+1)} - 3\ \bar{B}_{3}^{(2m+1)}\right] \qquad (138)$$

$$\mathcal{J}_{2r+1}^{(2m+1)} = \tfrac{1}{2}\left[\ (2r-1)\ \bar{B}_{2r-1}^{(2m+1)} - (2r+3)\ \bar{B}_{2r+3}^{(2m+1)}\right],\ r = 1,2,\ \cdots \qquad (139)$$

Using these expressions, the stresses (106), (107), and (108), may be written:

$$h^4\sigma_{11} = \sum_{m=o}^{\infty}\sum_{r=o}^{\infty}\left\{ D_{2m+1}\ {}^{11}Z_{2r+1}^{(2m+1)}(x_1) + F_{2m+1}\ {}^{11}V_{2r+1}^{(2m+1)}\ (x_1) + H_{2m+1}\ {}^{11}Q_{2r+1}^{(2m+1)}\ (x_1)\right\} \cos(2r+1)\ x_2\ e^{i(kx_3-wt)} \qquad (140)$$

$$\frac{h^4}{\mu}\ \sigma_{12} = \sum_{m=o}^{\infty}\sum_{r=o}^{\infty}\left\{ D_{2m+1}\ {}^{12}Z_{2r+1}^{(2m+1)}(x_1) + F_{2m+1}\ {}^{12}V_{2r+1}^{(2m+1)}(x_1) + H_{2m+1}\ {}^{12}Q_{2r+1}^{(2m+1)}(x_1)\right\} \sin\ (2r+1)x_2\ e^{i(kx_3-wt)} \qquad (141)$$

$$\frac{h}{\mu}\ \sigma_{13} = \sum_{m=o}^{\infty}\sum_{r=o}^{\infty}\left\{ D_{2m+1}\ {}^{13}Z_{2r+1}^{(2m+1)}(x_1) + F_{2m+1}\ {}^{13}V_{2r+1}^{(2m+1)}\ (x_1) + H_{2m+1}\ {}^{13}Q_{2r+1}^{(2m+1)}(x_1)\right\} \cos(2r+1)x_2\ e^{i(kx_3-wt)} \qquad (142)$$

where
$${}^{11}Z_{2r+1}^{(2m+1)}\ (x_1) = \left[-\frac{\omega^2\lambda f^4}{4c_d^2}\ (\tfrac{1}{2} + \cosh^2 2x_1)A_{2r+1}^{(2m+1)}\right.$$

$$-\frac{\omega^2\lambda f^4}{8c_d^2}\ W_{2r+1}^{(2m+1)} + \frac{\omega^2\lambda f^4}{2c_d^2}\ \cosh\ 2x_1\ U_{2r+1}^{(2m+1)}$$

$$+ \mu f^2 \cosh 2x_1 \left(a_{2m+1} - 2q_1 \cosh 2x_1\right) A_{2r+1}^{(2m+1)}$$

$$- \mu f^2 \left(a_{2m+1} - 2q_1 \cosh 2x_1\right) U_{2r+1}^{(2m+1)}$$

$$\left. - \mu f^2 \Gamma_{2r+1}^{(2m+1)} \right] Ce_{2m+1}(x_1, q_1) -$$

$$- \mu f^2 \sinh 2x_1 A_{2r+1}^{(2m+1)} \left[\frac{d}{dx_1} Ce_{2m+1} (x_1, q_1)\right] \tag{143}$$

$${}^{12}Z_{2r+1}^{(2m+1)}(x_1) = f^2 \left[\gamma_{2r+1}^{(2m+1)} - (2r+1)\cosh 2x_1 A_{2r+1}^{(2m+1)} - O_{2r+1}^{(2m+1)} \right] \left[\frac{d}{dx_1} Ce_{2m+1}(x_1, q_1)\right]$$

$$+ f^2 (2r+1) \sinh 2x_1 A_{2r+1}^{(2m+1)} Ce_{2m+1} (x_1, q_1) \tag{144}$$

$${}^{13}Z_{2r+1}^{(2m+1)} (x_1) = 2ik A_{2r+1}^{(2m+1)} \left[\frac{d}{dx_1} Ce_{2m+1} (x_1, q_1)\right] \tag{145}$$

$${}^{11}V_{2r+1}^{(2m+1)}(x_1) = \mu i k f^2 \left\{ \left[\left(\overline{a}_{2m+1} - 2q_2 \cosh 2x_1\right)\left(\cosh 2x_1 \overline{A}_{2r+1}^{(2m+1)} + \overline{U}_{2r+1}^{(2m+1)}\right) \right. \right.$$

$$\left.\left. + \overline{\Gamma}_{2r+1}^{(2m+1)} \right] Ce_{2m+1} (x_1, q_2) - \sinh 2x_1 \overline{A}_{2r+1}^{(2m+1)} \left[\frac{d}{dx_1} Ce_{2m+1}(x_1, q_2)\right] \right\} \tag{146}$$

$${}^{12}V_{2r+1}^{(2m+1)} (x_1) = ikf^2 \left\{ \left[\overline{\gamma}_{2r+1}^{(2m+1)} - (2r+1) \cosh 2x_1 \overline{A}_{2r+1}^{(2m+1)} - \overline{O}_{2r+1}^{(2m+1)} \right] \right.$$

$$\left. \times \left[\frac{d}{dx_1} Ce_{2m+1} (x_1, q_2)\right] + (2r+1) \sinh 2x_1 \overline{A}_{2r+1}^{(2m+1)} Ce_{2m+1}(x_1, q_2) \right\} \tag{147}$$

$${}^{13}V_{2r+1}^{(2m+1)} (x_1) = \left(\frac{\omega^2}{c_s^2} - 2k^2\right) \overline{A}_{2r+1}^{(2m+1)} \left[\frac{d}{dx_1} Ce_{2m+1} (x_1, q_2)\right] \tag{148}$$

$$ {}^{11}Q_{2r+1}^{(2m+1)}(x_1) = \frac{\mu\omega^2 f^2}{c_s^2}\left\{\left[(2r+1)\cosh 2x_1\,\overline{B}_{2r+1}^{(2m+1)} - \overline{\Lambda}_{2r+1}^{(2m+1)} - \overline{R}_{2r+1}^{(2m+1)}\right]\right. $$

$$ \left.\times\left[\frac{d}{dx_1}\,Se_{2m+1}(x_1,q_2)\right] - (2r+1)\sinh 2x_1\,\overline{B}_{2r+1}^{(2m+1)}\,Se_{2m+1}(x_1,q_2)\right\} \quad (149) $$

$$ {}^{12}Q_{2r+1}^{(2m+1)}(x_1) = \frac{\omega^2 f^2}{2c_s^2}\left\{\left[-(\overline{b}_{2m+1} - 2q_2\cosh 2x_1)(\cosh 2x_1\,\overline{B}_{2r+1}^{(2m+1)} + \overline{\epsilon}_{2r+1}^{(2m+1)})\right.\right. $$

$$ \left. + \overline{\delta}_{2r+1}^{(2m+1)} - (2r+1)^2\cosh 2x_1\,\overline{B}_{2r+1}^{(2m+1)} - 2\overline{d}_{2r+1}^{(2m+1)}\right] Se_{2m+1}(x_1,q_2) $$

$$ \left. + 2\sinh 2x_1\,\overline{B}_{2r+1}^{(2m+1)}\left[\frac{d}{dx_1}\,Se_{2m+1}(x_1,q_2)\right]\right\} \quad (150) $$

$$ {}^{13}Q_{2r+1}^{(2m+1)}(x_1) = \frac{\omega^2}{c_s^2}\,ik\,(2r+1)\,\overline{B}_{2r+1}^{(2m+1)}\,Se_{2m+1}(x_1,q_2) \quad (151) $$

The boundary conditions for the problem at hand are

$$ \sigma_{11} = \sigma_{12} = \sigma_{13} = 0,\ x_1 = \xi $$

where, as before, $x_1 = \xi$ denotes the boundary of the elliptical rod. Applying these conditions to (140), (141), and (142), gives, on interchanging the order of summation:

$$ \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\left\{D_{2m+1}\,{}^{11}Z_{2r+1}^{(2m+1)}(\xi) + F_{2m+1}\,{}^{11}V_{2r+1}^{(2m+1)}(\xi)\right. $$

$$ \left. + H_{2m+1}\,{}^{11}Q_{2r+1}^{(2m+1)}(\xi)\right\}\cos(2r+1)x_2 = 0 \quad (152) $$

$$ \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\left\{D_{2m+1}\,{}^{12}Z_{2r+1}^{(2m+1)}(\xi) + F_{2m+1}\,{}^{12}V_{2r+1}^{(2m+1)}(\xi)\right. $$

$$ \left. + H_{2m+1}\,{}^{12}Q_{2r+1}^{(2m+1)}(\xi)\right\}\sin(2r+1)x_2 = 0 \quad (153) $$

$$\sum_{r=o}^{\infty}\sum_{m=o}^{\infty}\left\{ D_{2m+1}\ {}^{13}Z_{2r+1}^{(2m+1)}(\xi) + F_{2m+1}\ {}^{13}V_{2r+1}^{(2m+1)}(\xi) + H_{2m+1}\ {}^{13}Q_{2r+1}^{(2m+1)}(\xi) \right\} \cos\,(2r+1)\,x_2 = 0 \tag{154}$$

Using the linear independence of the trigonometric functions, (152), (153), and (154), yield an infinite set of equations for the infinite number of unknowns D_{2m+1}, F_{2m+1}, and H_{2m+1}, $m = 0, 1, 2, \cdots$, and for these equations to have a solution, the determinant of the coefficients must equal zero. Thus the frequency equation is of the form:

$$\begin{vmatrix} \Delta_{11} & \Delta_{13} & \cdots \\ \Delta_{31} & \Delta_{33} & \cdots \\ \vdots & \vdots & \end{vmatrix} = 0 \tag{155}$$

where

$$\Delta_{11} = \begin{vmatrix} {}^{11}Z_1^{(1)}(\xi) & {}^{11}V_1^{(1)}(\xi) & {}^{11}Q_1^{(1)}(\xi) \\ {}^{12}Z_1^{(1)}(\xi) & {}^{12}V_1^{(1)}(\xi) & {}^{12}Q_1^{(1)}(\xi) \\ {}^{13}Z_1^{(1)}(\xi) & {}^{13}V_1^{(1)}(\xi) & {}^{13}Q_1^{(1)}(\xi) \end{vmatrix} \tag{156}$$

$$\Delta_{13} = \begin{vmatrix} {}^{11}Z_1^{(3)}(\xi) & {}^{11}V_1^{(3)}(\xi) & {}^{11}Q_1^{(3)}(\xi) \\ {}^{12}Z_1^{(3)}(\xi) & {}^{12}V_1^{(3)}(\xi) & {}^{12}Q_1^{(3)}(\xi) \\ {}^{13}Z_1^{(3)}(\xi) & {}^{13}V_1^{(3)}(\xi) & {}^{13}Q_1^{(3)}(\xi) \end{vmatrix} \tag{157}$$

$$\Delta_{31} = \begin{vmatrix} {}^{11}Z_3^{(1)}(\xi) & {}^{11}V_3^{(1)}(\xi) & {}^{11}Q_3^{(1)}(\xi) \\ {}^{12}Z_3^{(1)}(\xi) & {}^{12}V_3^{(1)}(\xi) & {}^{12}Q_3^{(1)}(\xi) \\ {}^{13}Z_3^{(1)}(\xi) & {}^{13}V_3^{(1)}(\xi) & {}^{13}Q_3^{(1)}(\xi) \end{vmatrix} \tag{158}$$

$$\Delta_{33} = \begin{vmatrix} {}^{11}Z_3^{(3)}(\xi) & {}^{11}V_3^{(3)}(\xi) & {}^{11}Q_3^{(3)}(\xi) \\ {}^{12}Z_3^{(3)}(\xi) & {}^{12}V_3^{(3)}(\xi) & {}^{12}Q_3^{(3)}(\xi) \\ {}^{13}Z_3^{(3)}(\xi) & {}^{13}V_3^{(3)}(\xi) & {}^{13}Q_3^{(3)}(\xi) \end{vmatrix} \tag{159}$$

As in the previous example, the question of the elliptical rod approaching the circular rod will now be examined. To illustrate the process, the term ${}^{11}Z_1^{(1)}(\xi)$ will be examined in detail. To this end it should be noted that (a) as the eccentricity e goes to zero the coefficients $A_{2r+1}^{(2m+1)}$, $\overline{A}_{2r+1}^{(2m+1)}$, and $\overline{B}_{2r+1}^{(2m+1)}$, go to one, for $r = m$, and go to zero for $r \neq m$ (b) as e goes to zero it follows from (41) that f goes to zero and x_1 (i.e., ξ in the above) goes to infinity (c) a_{2m+1} and b_{2m+1} go to $(2m+1)^2$ as q (i.e., e) goes to zero (see [24], p. 724). Using these facts it follows from (57), (64), (102), (110), (113), and (117), and the fact that the semi-major axis a and the semi-minor axis b now approach the circular rod radius (a, say) that

$$\lim_{e\to 0} {}^{11}Z_1^{(1)}(\xi) = \lim_{e\to 0} \left\{ \left[-\frac{\omega^2 \lambda a^4 \cosh^2 2\xi}{4\, c_d^2 \cosh^4\xi} + \frac{\mu a^2 \cosh 2\xi}{\cosh^2\xi} \left(1 - \frac{2a^2 p_1^2 \cosh 2\xi}{4 \cosh^2 \xi} \right) \right] Ce_1(\xi, q_1) - \frac{\mu a^2 \sinh 2\xi}{\cosh^2 \xi} \left[\frac{d}{d\xi} Ce_1(\xi, q_1) \right] \right\}$$

$$= a^4 k_1 \left\{ \left[-\frac{\omega^2 \lambda}{c_d^2} + 2\mu \left(\frac{2}{a^2} - p_1^2 \right) \right] J_1(p_1 a) - \frac{2\mu}{a} p_1 J_o(p_1 a) \right\}$$

Similarly it can be shown that:

$$\lim_{e \to o} {}^{12}Z_1^{(1)}(\xi) = 2k_1 a^4 \left[\frac{2}{a^2} J_1(p_1 a) - \frac{p_1}{a} J_o(p_1 a) \right]$$

$$\lim_{e \to o} {}^{13}Z_1^{(1)}(\xi) = 2ikk_1 a \left[p_1 J_o(p_1 a) - \frac{1}{a} J_1(p_1 a) \right]$$

$$\lim_{e \to o} {}^{11}V_1^{(1)}(\xi) = 2\mu ikk_1 a^4 \left[\left(\frac{2}{a^2} - p_2^2\right) J_1(p_2 a) - \frac{p_2}{a} J_o(p_2 a) \right]$$

$$\lim_{e \to o} {}^{12}V_1^{(1)}(\xi) = 2ikk_1 a^4 \left[\frac{2}{a^2} J_1(p_2 a) - \frac{p_2}{a} J_o(p_2 a) \right]$$

$$\lim_{e \to o} {}^{13}V_1^{(1)}(\xi) = k_1 \left(\frac{\omega^2}{c_s^2} - 2k^2\right) a \left[p_2 J_o(p_2 a) - \frac{1}{a} J_1(p_2 a) \right]$$

$$\lim_{e \to o} {}^{11}Q_1^{(1)}(\xi) = \frac{2\mu\omega^2 a^4 k_1'}{c_s^2} \left[\frac{p_2}{a} J_o(p_2 a) - \frac{2}{a^2} J_1(p_2 a) \right]$$

$$\lim_{e \to o} {}^{12}Q_1^{(1)}(\xi) = \frac{\omega^2 k_1' a^4}{c_s^2} \left[\left(p_2^2 - \frac{4}{a^2}\right) J_1(p_2 a) + \frac{2p_2}{a} J_o(p_2 a) \right]$$

$$\lim_{e \to o} {}^{13}Q_1^{(1)}(\xi) = ikk_1' \frac{\omega^2}{c_s^2} J_1(p_2 a)$$

Hence it is readily seen that Δ_{11}, as given by (156), set equal to zero reduces to the frequency equation (39) for the propagation of harmonic flexural waves in an infinite circular rod of radius a with stress-free boundaries. In a similar fashion it can be shown that as the eccentricity e goes to zero the off-diagonal terms in (155) go to zero and Δ_{33} goes to the frequency equation for the propagation of a higher order circumferential mode in a circular rod of radius a, with stress-free boundaries. Thus, as in the previous example, the frequency equation reduces to a product of circular rod frequency equations, each describing the propagation of a circumferential mode.

The other possible modes of propagation, i.e., the other "flexural" type, the "compressional" type, and the "torsional" type, can be treated in the same fashion and frequency equations in the form of infinite determinants are obtained in each case.

Though the theory of infinite determinants is well-developed (see, for instance, Whittaker and Watson [25] § 2.8), very little information as to possible numerical procedures seems to be available. Taylor [26], in connection with the buckling of a rectangular plate, evaluates an infinite determinant by examining the convergence of a sequence of n x n determinants, $n = 2,3,4, \cdots$, taking the upper left hand corner of the infinite determinant as the starting point. This procedure could be adopted in the present case also, but in general the problem appears to be quite formidable. However the approach does look feasible for small values of the eccentricity e. In the present work the first determinant of the sequence should be the 3x3 determinant Δ_{11}, as given by either (98) or (156), since it is known that this describes the basic physical process in the circular rod case. Setting this equal to zero enables one to obtain an ω-k relationship. The next determinant in the sequence is a 6x6 determinant and inspection of (97) and §3.33 in Reference [21] shows that this set equal to zero gives a relationship of the form $\Delta_{11} = O(e^4)$, so that the correction term is small for small values of e.

This approach, which looks quite promising, will be examined in greater detail in a later work. In connection with any numerical evaluation, the increasing appearance of new tables of Mathieu functions should be mentioned (see [24] for a bibliography of tables and Barakat, Haustor, and Levin [27] for later work).

Finally, mention will be made of other problems to which the above techniques are applicable. The infinite elliptical shell could be treated in the same fashion, the only difference being that in this case the second linearly independent solutions to (55), (56), etc., must be retained. Similar type of work could be done for cylinders formed by parabolic arcs, but the numerical difficulties are even more severe in this case, because of the limited tabulation of the parabolic cylindrical functions. In this connection Nakonechny and Callahan [28], using the Mindlin equations, obtained frequency equations in the form of infinite determinants for plates formed by parabolic arcs. Of possible interest also is the question of wave propagation in a parabolic wedge, since in that case the problem of a sharp edge does not arise.

APPENDIX A

SOLUTION OF THE VECTOR WAVE EQUATION IN TERMS OF SCALAR POTENTIALS

In this appendix it will be shown that the vector wave equation can be solved in terms of two scalar potentials, which satisfy scalar wave equations, for the case of (i) general cylindrical coordinates and (ii) spherical coordinates.

(1) GENERAL CYLINDRICAL COORDINATES

General cylindrical coordinates are defined by the orthogonal transform

$$\begin{aligned} x + iy &= w(\xi) = w(x_1 + ix_2) \\ z &= x_3 \end{aligned} \tag{A1}$$

where $\omega(\xi)$ is an analytic function of $\xi = x_1 + ix_2$. If $\psi(x_1,x_2,x_3,t)$ is a scalar function, then it can be readily shown that

(a)
$$\nabla \times (\overrightarrow{e_{x_3}}\,\psi) = \nabla\psi \times \overrightarrow{e_{x_3}} \tag{A2}$$

$$\nabla \times \nabla \times (\overrightarrow{e_{x_3}}\,\psi) = \nabla \frac{\partial\psi}{\partial x_3} - \overrightarrow{e_{x_3}}\nabla^2\psi \tag{A3}$$

where $\overrightarrow{e_{x_3}}$ is a unit vector in the x_3-direction.

(b) If $\psi(x_1,x_2,x_3,t)$ satisfies the scalar wave equation

$$\nabla^2 \psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2} \tag{A4}$$

then substitution of (A4) into (A3) gives

$$\nabla \times \nabla \times (\vec{e_{x_3}}\,\psi) = \nabla\, \frac{\partial \psi}{\partial x_3} - \vec{e_{x_3}}\,\frac{1}{c_s^2}\frac{\partial^2 \psi}{\partial t^2} \tag{A5}$$

(c) If ψ satisfies $\nabla^2 \psi = \frac{1}{c_s^2}\frac{\partial^2 \psi}{\partial t^2}$ (A6)

and χ satisfies

$$\nabla^2 \chi = \frac{1}{c_s^2}\frac{\partial^2 \chi}{\partial t^2} \tag{A7}$$

then

$$\vec{A} = \nabla \times (\vec{e_{x_3}}\,\psi) + \nabla \times \nabla \times (\vec{e_{x_3}}\,\chi) \tag{A8}$$

satisfies

$$\nabla^2 \vec{A} = \frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2}\,. \tag{A9}$$

<u>Proof</u>

Taking the divergence of (A8) gives

$$\nabla \cdot \vec{A} = 0 \tag{A10}$$

Taking the curl of (A8) gives, on using (A5),

$$\begin{aligned}\nabla \times \vec{A} &= \nabla \times \nabla \times (\vec{e_{x_3}}\,\psi) + \nabla \times \left[\nabla \left(\frac{\partial \chi}{\partial x_3}\right) - \vec{e_{x_3}}\,\frac{1}{c_s^2}\frac{\partial^2 \chi}{\partial t^2}\right] \\ &= \nabla \left(\frac{\partial \psi}{\partial x_3}\right) - \vec{e_{x_3}}\,\frac{1}{c_s^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{c_s^2}\nabla \times \left(\vec{e_{x_3}}\,\frac{\partial^2 \chi}{\partial t^2}\right)\end{aligned} \tag{A11}$$

Taking the curl of (A11) gives, on using (A8) and (A10),

$$\begin{aligned}\nabla \times \nabla \times \vec{A} &= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = -\nabla^2 \vec{A} \\ &= \nabla \times \left[\nabla \left(\frac{\partial \psi}{\partial x_3}\right) - \vec{e_{x_3}}\,\frac{1}{c_s^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{c_s^2}\nabla \times \left(\vec{e_{x_3}}\,\frac{\partial^2 \chi}{\partial t^2}\right)\right]\end{aligned}$$

$$= -\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\left[\nabla \times (\overrightarrow{e_{x_3}}\,\psi) + \nabla \times \nabla \times (\overrightarrow{e_{x_3}}\,\chi)\right]$$

$$= -\frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} \qquad \text{(A12)}$$

Hence
$$\nabla^2 \vec{A} = \frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} \qquad \text{(A13)}$$

(d) If (A6) and (A7) are satisfied, then

$$\vec{A} = \overrightarrow{e_{x_3}}\,\psi + \nabla \times (\overrightarrow{e_{x_3}}\,\chi) \qquad \text{(A14)}$$

will also satisfy
$$\nabla^2 \vec{A} = \frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} \, .$$

<u>Proof</u>

Taking the divergence of (A14) gives

$$\nabla \cdot \vec{A} = \nabla \cdot (\overrightarrow{e_{x_3}}\,\psi) = \frac{\partial \psi}{\partial x_3} \qquad \text{(A15)}$$

Taking the curl of (A14) gives, on using (A5),

$$\nabla \times \vec{A} = \nabla \times \left[\overrightarrow{e_{x_3}}\,\psi + \nabla \times (\overrightarrow{e_{x_3}}\,\chi)\right]$$

$$= \nabla \times (\overrightarrow{e_{x_3}}\,\psi) + \nabla\left(\frac{\partial \chi}{\partial x_3}\right) - \overrightarrow{e_{x_3}}\,\frac{1}{c_s^2}\frac{\partial^2 \chi}{\partial t^2} \qquad \text{(A16)}$$

Taking the curl of (A16) gives, on using (A3), (A14), and (A15),

$$\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$= \nabla \times \left[\nabla \times (\psi \overrightarrow{e_{x_3}}) + \nabla \left(\frac{\partial \chi}{\partial x_3}\right) - \overrightarrow{e_{x_3}} \frac{1}{c_s^2} \frac{\partial^2 \chi}{\partial t^2} \right]$$

$$= \nabla \left(\frac{\partial \psi}{\partial x_3}\right) - \overrightarrow{e_{x_3}} \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla \times \left(\overrightarrow{e_{x_3}} \frac{1}{c_s^2} \frac{\partial^2 \chi}{\partial t^2}\right)$$

$$= \nabla (\nabla \cdot \vec{A}) - \frac{1}{c_s^2} \frac{\partial^2 \vec{A}}{\partial t^2} \tag{A17}$$

Hence $\nabla^2 \vec{A} = \frac{1}{c_s^2} \frac{\partial^2 \vec{A}}{\partial t^2}$ (A18)

(ii) <u>SPHERICAL COORDINATES</u>

If $\psi(r,\theta,\varphi,t)$ is a scalar function of the spherical coordinates r,θ,φ, then it can be readily shown that:

(a)
$$\nabla \times (\overrightarrow{e_r}\, r\, \psi) = (\nabla r\, \psi) \times \overrightarrow{e_r} \tag{A19}$$

$$\nabla \times \nabla \times (\overrightarrow{e_r}\, r\, \psi) = \nabla \left(\frac{\partial r \psi}{\partial r}\right) - \overrightarrow{e_r}\, r\, \nabla^2 \psi \tag{A20}$$

where $\overrightarrow{e_r}$ is a unit vector in the r-direction.

(b) If $\psi(r,\theta,\varphi,r)$ satisfies the scalar wave equation

$$\nabla^2 \psi = \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} \tag{A21}$$

then substitution of (A21) into (A20) gives

$$\nabla \times \nabla \times (\overrightarrow{e_r}\, r\, \psi) = \nabla \left(\frac{\partial r \psi}{\partial r}\right) - \overrightarrow{e_r}\, r\, \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} \tag{A22}$$

(c) If ψ satisfies $\nabla^2 \psi = \frac{1}{c_s^2} \frac{\partial \psi}{\partial t^2}$ (A23)

and χ satisfies
$$\nabla^2 \chi = \frac{1}{c_s^2} \frac{\partial^2 \chi}{\partial t^2} \tag{A24}$$

then
$$\begin{aligned} \vec{A} &= \nabla \times (\vec{e_r}\, r\, \psi) + \nabla \times \nabla \times (\vec{e_r}\, r\, \chi) \\ &= \nabla \times (\vec{e_r}\, r\, \psi) + \nabla \left(\frac{\partial r\chi}{\partial r}\right) - \vec{e_r}\, \frac{1}{c_s^2} \frac{\partial^2 r\chi}{\partial t^2} \end{aligned} \tag{A25}$$

satisfies
$$\nabla^2 \vec{A} = \frac{1}{c_s^2} \frac{\partial^2 \vec{A}}{\partial t^2} \tag{A26}$$

<u>Proof</u>

Taking the divergence of (A25) gives

$$\nabla \cdot \vec{A} = 0 \tag{A27}$$

Taking the curl of (A25) gives, on using (A22),

$$\begin{aligned} \nabla \times \vec{A} &= \nabla \times \left[\nabla \times (\vec{e_r}\, r\, \psi) + \nabla \left(\frac{\partial r\chi}{\partial r}\right) - \vec{e_r}\, \frac{1}{c_s^2} \frac{\partial^2 r\chi}{\partial t^2} \right] \\ &= \nabla \left(\frac{\partial r\psi}{\partial r}\right) - \vec{e_r}\, \frac{1}{c_s^2} \frac{\partial^2 r\psi}{\partial t^2} - \nabla \times \left(\vec{e_r}\, \frac{1}{c_s^2} \frac{\partial^2 r\chi}{\partial t^2} \right) \end{aligned} \tag{A28}$$

Taking the curl of (A28) gives, on using (A27) and (A25),

$$\begin{aligned} \nabla \times \nabla \times \vec{A} &\equiv \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = - \nabla^2 \vec{A} \\ &= \nabla \times \left[\nabla \left(\frac{\partial r\psi}{\partial r}\right) - \vec{e_r}\, \frac{1}{c_s^2} \frac{\partial^2 r\chi}{\partial t^2} - \nabla \times \left(\vec{e_r}\, \frac{1}{c_s^2} \frac{\partial^2 r\chi}{\partial t^2} \right) \right] \\ &= - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \left[\nabla \times (\vec{e_r}\, r\, \psi) + \nabla \times \nabla \times (\vec{e_r}\, r\, \chi) \right] = - \frac{1}{c_s^2} \frac{\partial^2 \vec{A}}{\partial t^2} \end{aligned}$$

i.e.,

$$\nabla \times \nabla \times \vec{A} = -\frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

Hence

$$\nabla^2 \vec{A} = \frac{1}{c_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} .$$

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